EE106A Discussion 2: Exponential Coordinates

1 Rigid Body Transformations

Problem 1: Recall that a mapping $G : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be a rigid body transformation if it satisfies the following properties:

1. Length preservation: $\forall$ points $p, q \in \mathbb{R}^3$, $||p - q|| = ||g(p) - g(q)||$

2. Orientation preservation: $\forall$ vectors $v, w \in \mathbb{R}^3$, $g(v \times \omega) = g(v) \times g(\omega)$

Rotations, translations, and combined rotations+translations are all rigid body transformations!

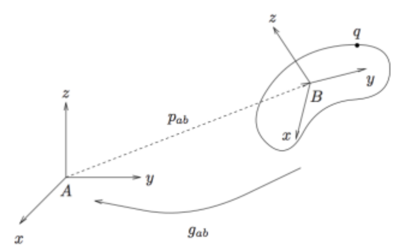


Figure 1: A rigid body transformation.

1.1 Rigid Transformation of a Point

We can move and rotate coordinate frames. Points on that frame move and rotate with it!

Exercise: *Write out the equation for an affine rigid body transformation of a point. Apply this to a robot arm that has rotated π radians about the y-axis and translated 1 unit in the y-direction. Find the new location of a sensor originally located at $[2, 2, 2]^T$.*

$$q_a = p_{ab} + R_{ab} q_b$$
$$g(q) = p + R(q)$$

$$R_{AB} = R_y(\pi) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$t_{AB} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$p' = R_{AB}p + t_{AB}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

1.2 Rigid Transformation of a Vector

Recall that vectors do not have position, only orientation.

Exercise: *How can we modify the rigid body transformation to apply to vectors?*
We just include the rotational component of the rigid body transformation:

$$g(v) = g(s - r) = g(s) - g(r) = (p + R(s)) - (p + R(r)) = R(s) - R(r) = R(s - r) = R(v)$$

1.3 Homogeneous Coordinates

Homogeneous coordinates allow us to express transformations using a single matrix, just like rotations. They are 4-dimensional vectors that provide a distinction between points and vectors.

Exercise: *Write out the homogeneous coordinates for a point and a vector.*

$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

1.4 Homogeneous Transformation Matrices

Now, let's find out the matrix that allows us to express rigid body transformations of both points and vectors!

Exercise: *Write out the homogeneous transformation matrix, and show how it will transform the homogeneous coordinates of both points and vectors correctly.*

$$G_{AB} = \begin{bmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{bmatrix}$$

$$G_{AB}p_B = \begin{bmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_B \\ 1 \end{bmatrix} = R_{AB}p_B + t_{AB} = \begin{bmatrix} p' \\ 1 \end{bmatrix}$$

$$G_{AB}v_B = \begin{bmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_B \\ 0 \end{bmatrix} = R_{AB}v_B + 0 = \begin{bmatrix} v' \\ 0 \end{bmatrix}$$

Exercise: *Write out the above transformation of the robot arm using a homogeneous transformation matrix.*

$$p' = G_{AB}p$$

$$p' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

This retains the homogeneous coordinate of the point and correctly transforms it!

1.5 Composition Rule

Homogeneous transformation matrices can be composed just like rotation matrices!

Exercise: *Given G_{AB} and G_{BC}, find G_{AC}.*

$$G_{AC} = G_{AB}G_{BC}$$

1.6 Inverse

The inverse of a homogeneous transformation matrix is given by

$$G^{-1} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$

2 Exponential Coordinates

The matrix exponential of A, e^A, is defined to be

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ...$$

Also recall that the solution to the differential equation

$$\dot{x} = Ax$$

is given by

$$x(t) = e^{At}x_0$$

2.1 Motivation

We want to construct transformation matrices (either pure rotation or homogeneous) to understand how our robot is moving. Robots, however, have continuous motion - we can't just express that with a single matrix. We want our matrix to be some function of θ (how much our arm has moved).

How can we make our rotation matrices to be a function of time or θ? We can look at how the joint moves! Let's see how that happens in practice...

2.2 Exponential Coordinates for Rotation

It turns out that we can use the idea of the matrix exponential to calculate rotation matrices parameterized by time! (We'll get to full homogeneous transformation matrices next)

In order to do so, we need our exponential coordinates:

- $\omega \in \mathbb{R}^3$: our axis of rotation
- θ: the extent of rotation (scalar)

Exercise: *Find the rotation matrix $R(\omega, \theta)$ for a rotation about some axis ω by amount θ. How is Rodrigues' formula related?*

In this formulation, ω is a vector that passes through the origin. We let point p be a point that rotates about ω with angular unit velocity, so $||\omega|| = 1$.

The velocity of this point is

$$\begin{aligned}\dot{p}(t) &= \omega \times p(t) \\ &= \widehat{\omega} p(t)\end{aligned}$$

Solving for this linear differential equation, we have

$$p(t) = e^{\widehat{\omega} t} p(0)$$

where $p(0)$ is the initial position of the point.

Since we constructed the angular velocity to be of unit magnitude, we can reparameterize by replacing t with θ in the solution, so

$$p(\theta) = e^{\widehat{\omega}\theta} p(0)$$

So, our desired rotation matrix is

$$R(\omega, \theta) = e^{\widehat{\omega}\theta}$$

Recall that Rodrigues' formula gives us a simpler way to calculate this matrix exponential:

$$e^{\widehat{\omega}\theta} = I + \widehat{\omega} sin\theta + \widehat{\omega}^2(1 - cos\theta)$$

Exercise: *Find the exponential coordinates (ω, θ) of the rotation matrix $R_y(\frac{\pi}{2})$.*

$$\omega = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \theta = \frac{\pi}{2}$$

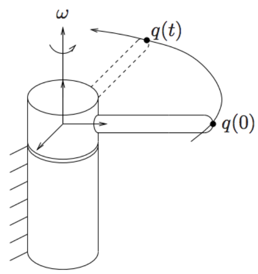


Figure 2: Rotations can happen about any arbitrary axis ω. In this figure the ω axis appears to be coincident with the z-axis, but it can actually be any general vector!

3 Exponential coordinates for rigid motion

We usually want to find more than just the rotation matrix - we want to see how position changes as well. As a result, we want to use the same idea of exponential coordinates to construct parameterized versions of the full homogeneous transformation matrix!

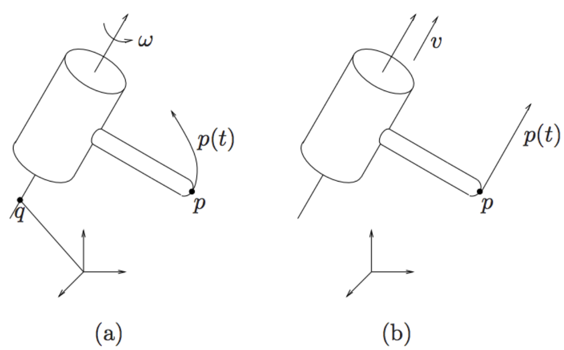


Figure 3: a) A revolute joint and b) a prismatic joint.

Most modern robotic rigid body transformations are enabled by *joints*, which connect sets of rigid *links* together. In this class, we focus on two types of joints — *revolute* and *prismatic* joints (see Fig. 3). Revolute joints allow adjacent links to rotate relative to each other about a fixed axis, and prismatic joints allow links to move linearly relative to each other along a fixed axis.

Exercise: *Write the expressions for the velocity of the point p (ie. $\dot{p}(t)$) when attached to the revolute joint and attached to the prismatic joint in Fig. 3. Assume that $\omega \in \mathbb{R}^3$, $||\omega|| = 1$, and $q \in \mathbb{R}^3$ is some point along the axis of ω.*

For the revolute joint: $\dot{p}(t) = \omega \times (p(t) - q)$

For the prismatic joint: $\dot{p}(t) = v$

3.1 Twist of revolute joint

Now, let's make the above velocity into a differential equation in homogeneous coordinates. Recall that in homogeneous coordinates, append a 0 to vectors and a 1 to points.

Exercise: *Find $\widehat{\xi}$ to complete the following expression of $\dot{p}(t)$ in homogeneous coordinates for a revolute joint.*

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \widehat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix}}_{=:\, \widehat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Hint: Recall the skew symmetric matrix $\widehat{\omega}$ of ω, which allows you to take a cross product with ω:

$$\omega = [\omega_1 \;\; \omega_2 \;\; \omega_3]^T; \;\; \widehat{w} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{1}$$

3.2 Twist of prismatic joint

Exercise *Find $\widehat{\xi}$ to complete the following expression of $\dot{p}(t)$ in homogeneous coordinates for a prismatic joint.*

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}}_{=:\, \widehat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

3.3 Vee and wedge operators of a twist

The above quantity we have derived, $\widehat{\xi}$, is called a *twist.* A twist captures the angular and linear velocities of a body. There are two handy operators that we use on twist entities. First, given a twist $\widehat{\xi} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^4$, the $\vee$ (vee) operator extracts the 6-dimensional vector which parameterizes a twist, where $\xi := (v, \omega)$ are the *twist coordinates* of $\widehat{\xi}$.

$$\begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}^\vee = \begin{bmatrix} v \\ \omega \end{bmatrix} =: \xi \tag{2}$$

The inverse operator, $\wedge$ (wedge), constructs a matrix out of a vector of the twist coordinates:

$$\begin{bmatrix} v \\ \omega \end{bmatrix}^\wedge = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \tag{3}$$

Exercise: *Find the twist coordinates for a revolute and prismatic joint.*

Revolute:

$$\begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$

Prismatic:

$$\begin{bmatrix} v \\ 0 \end{bmatrix}$$

3.4 Solution to differential equation gives us the exponential map

Exercise: *Write the general solution to the differential equation* $\dot{\bar{p}} = \widehat{\xi}\bar{p}$*. Then, make use of the fact that* $||\omega|| = 1$ *to reparameterize* t *to be* θ*. Specifically, find the expression for* $p(\theta)$ *in terms of* $p(0)$*.*

$$\bar{p}(t) = e^{\widehat{\xi}t}\bar{p}(0)$$

$$\bar{p}(\theta) = e^{\widehat{\xi}\theta}\bar{p}(0)$$

This transformation is not the same as the rigid transformations we studied previously in that it is not a mapping from one coordinate frame to another, but rather the mapping of points from their initial coordinates $p(0)$ to their coordinates after a rigid motion parameterized by a joint angle θ is applied. It turns out that the matrix exponential simplifies to:

$$e^{\widehat{\xi}\theta} = \begin{bmatrix} e^{\widehat{\omega}\theta} & (I - e^{\widehat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} \tag{4}$$

Our exponential coordinates for general rigid body motion are (ξ, θ).

3.5 Screw motion

Chasles' Theorem states that any rigid body transformation can be decomposed into an equivalent finite rotation about a fixed axis and a finite translation along that same axis. This is what we call a *screw motion* S, which consists of an axis l, a pitch h, and a magnitude M. It is equivalent to a rotation by an amount $\theta = M$ about l followed by a translation by $h\theta = hM$ along l (see Fig. 3.5). ($h = 0$ corresponds to pure rotation, and $h = \infty$ corresponds to pure translation).

The transformation G corresponding to S has the following effect on a point p:

$$Gp = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega \tag{5}$$

Converted to matrix form, this would be

$$g\begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\widehat{\omega}\theta} & (I - e^{\widehat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p \\ 1 \end{bmatrix}$$

This is very similar in form to the expression in Eq. 4. This is just to build intuition that every twist corresponds to an equivalent screw motion and vice versa.

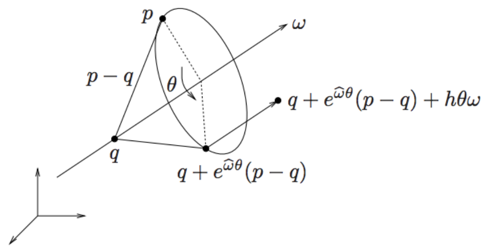
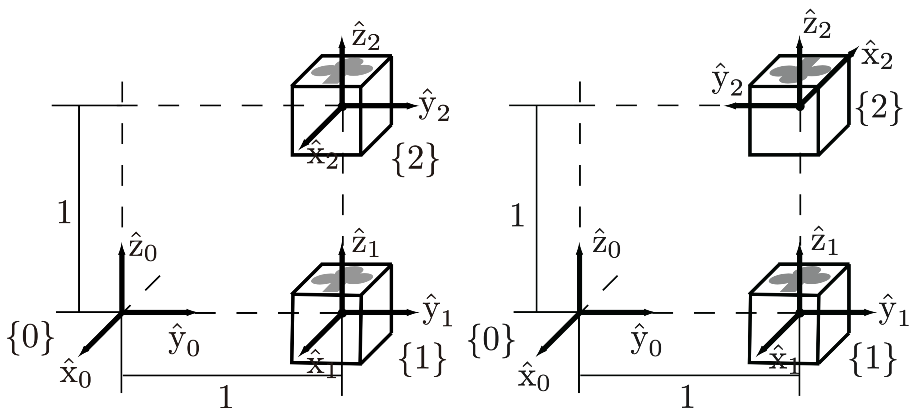


Figure 4: Generalized screw motion.

4 Finding Exponential Coordinates

Figure (4) shows a cube undergoing two different rigid body transformations from frame $\{1\}$ to frame $\{2\}$. In both cases, find a set of exponential coordinates for the rigid body transform that maps the cube from its initial to its final configuration, as viewed from frame $\{0\}$. Do this by first finding the equivalent screw motion.



(a) A first screw motion. (b) A second screw motion.

A cube undergoing two screw motions.

The first transformation is simply a translation along the axis $(0, 0, 1)^T$ by 1 unit as seen from frame $\{1\}$. This is a screw with infinite pitch and unit magnitude. So the required exponential coordinates are $(\xi = (v, \omega), \theta)$, where $v = (0, 0, 1)^T, \omega = 0, \theta = 1$.

The second transformation is likewise a translation along axis $\omega = (0, 0, 1)^T$ and a rotation by $\theta = \pi$ about an axis in the direction ω passing through $q = (0, 1, 0)$. This is a screw motion along axis $\omega = (0, 0, 1)^T$ by magnitude $\theta = \pi$ and pitch $h = 1/\pi$. Therefore the corresponding Twist is $\xi = (v, \omega)$ where

$$v = -\omega \times q + h\omega = (1, 0, 1/\pi)^T \tag{6}$$

and the exponential coordinates are then (ξ, θ) for $\theta = \pi$.