EE106A Discussion 4: Inverse Kinematics

1 Inverse kinematics

In forward kinematics, we found the expression for $g_{st}(\theta)$ as a function of θ. Now, in inverse kinematics, we are given a desired configuration of the tool frame g_d, and we wish to find the set of θs for which

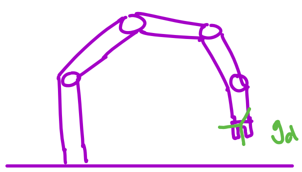
$$e^{\hat{\xi}_1\theta_1}...e^{\hat{\xi}_n\theta_n}g_{st}(0) = g_{st}(\theta) = g_d \tag{1}$$

Unknown

Known

Given:

- Desired configuration
  - We know where we want our **tool** to end up
  - Ex. In position to grab a box on the table



$g_d \in SE(3)$

- Also know details about the robot itself
  - I.e., we know the **twists** and **starting configuration**

$\xi_1 \ldots \xi_n$ $\qquad$ $g_{st}(0)$

Desired:

- How do we angle each individual joint to get us there?
  - Allow us to move the robot to position it properly
  - Find **thetas**

$\theta_1 \ldots \theta_n$

2 Padan-Kahan subproblems

To solve the inverse kinematics problem, one technique is to distill it into the following three simpler subproblems for which we know the solutions.

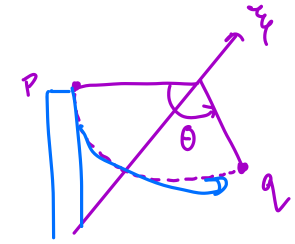
- We know the solutions to some **basic** inverse kinematics problems
  - If our problem is in the from of one of these basic ones, we can find theta
- Can we reduce the super complicated robot problem down to the basic ones?

* Only revolute axes

Subproblems Overview

- Subproblem 1
    - Rotate about some fixed axis
    - Pure rotation about axis

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ \omega \end{bmatrix}$$



$$e^{\hat{\xi}\theta} \cdot p = q$$

≤ 1 solution

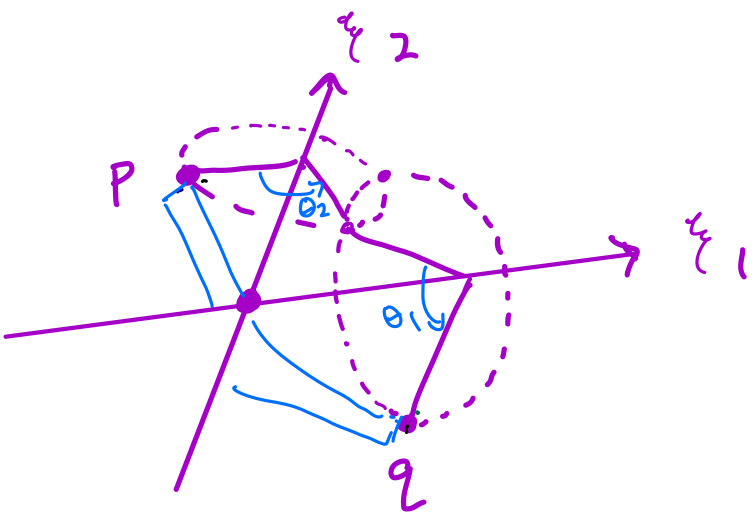
Need to specify p, q

$\boxed{\text{Find } \theta}$

$\theta = \mathrm{atan2}(w^T(u' \times v'), u' \cdot v')$

(won't need to ever compute this)

- Subproblem 2
    - Rotate about 2 intersecting axes



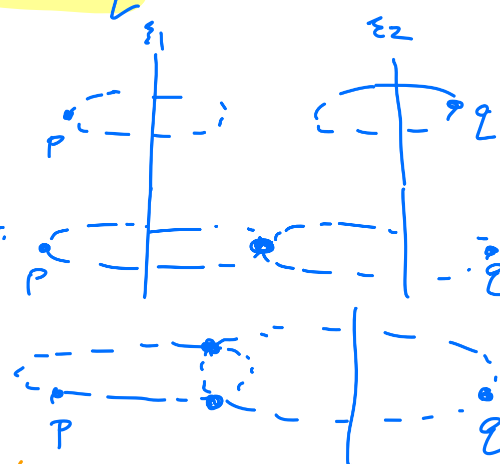
$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = q$$

$\boxed{\leq 2 \text{ sols}}$

0 solutions:

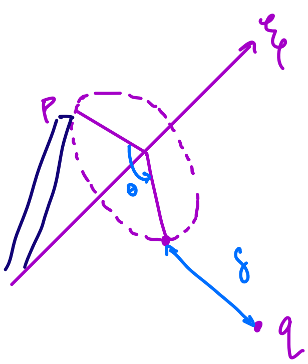
1 solution:

2 sols:



θ_1, θ_2 = * See textbook

- Subproblem 3
    - Move one point to a specified distance from another



$$\| e^{\hat{\xi}_1\theta_1} p - q \| = \delta$$

Distance btwn. rotated p & q $= \delta$

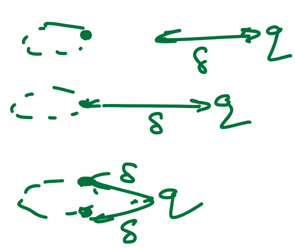
$\boxed{\leq 2 \text{ sols}}$

θ = * See worksheet

0:

1:

2:



Okay, so we know we can solve these subproblems. How does that help me with a large robot?

Great question. Our goal is to try to reduce the number of unknowns.

- Use specially chosen points
- Reduce the problem to only 1 or 2 unknown thetas
- Apply subproblems to solve for remaining variables

Tricks

$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0) = g \longrightarrow \text{FK solution}$$

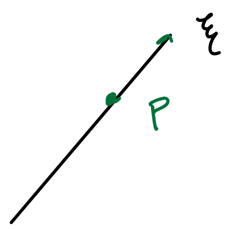
$\downarrow$

Rearrange $\longrightarrow$ $g_{st}(0)$ is known & invertible

$\downarrow$

$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} = g_d \, g_{st}^{-1}(0) := g_1$$

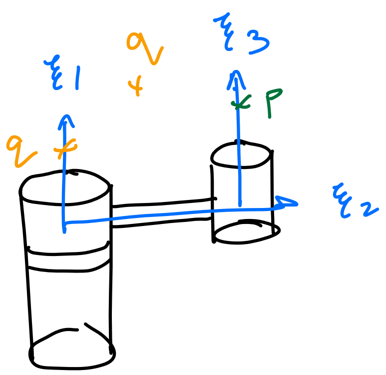
Trick #1: Choose a clever point (eliminate variables from RHS of our prod. of exps.**)**



$$\boxed{e^{\hat{\xi}\theta} p = p}$$

↳ p is on the axis

Rotation won't change location of p



$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \underbrace{e^{\hat{\xi}_3\theta_3} \cdot p} = g \cdot p, \quad g = g_d \cdot g_{st}^{-1}(0)$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = g \cdot p$$

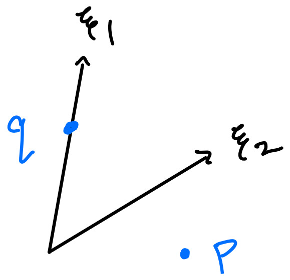
$\rightarrow$ Use SP2 to solve θ_1, θ_2

$$e^{\hat{\xi}_3\theta_3} = \overbrace{e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}}^{\text{Know, can invert}} \cdot g := g_1$$

$$e^{\hat{\xi}_3\theta_3} \cdot q = g_1 \cdot q \longrightarrow \boxed{\theta_3 \text{ w/ SP1}}$$

Trick 2: Subtract a point from both sides and take norm (eliminate variables from LHS)

of prod. of exp.



$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} = g$$

- q on axis of joint 1
- p <u>not</u> on axis of joint 2

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdot p - q = gp - q$$

q is on ξ_1

$$\Rightarrow e^{\hat{\xi}_1\theta_1}\left(e^{\hat{\xi}_2\theta_2} p - q\right) = gp - q$$

Magnitude

$$\| \underbrace{e^{\hat{\xi}_1\theta_1}}_{\text{Rigid body transform}} \underbrace{(e^{\hat{\xi}_2\theta_2} p - q)}_{\text{Vector}} \| = \| gp - q \|$$

Rigid transform preserves magnitude

$$\| e^{\hat{\xi}_2\theta_2} p - q \| = \underbrace{\| gp - q \|}_{\text{Known scalar} = \delta}$$

$$\boxed{\text{Use SP 3 to solve } \theta_2}$$

Trick 3: Prismatic Joints

- Good to solve these 1st
- Use geometry of robot

$\| e^{\hat{\xi}\theta} p - q \| = \delta$

l_1 = starting distance

Get to this form w/ trick 2:

$$\| e^{\hat{\xi}\theta} p - q \| = \delta$$

$$\delta = l_1 + \theta$$

$$\theta = \delta - l_1$$

$$\delta = l_1 + \theta$$

$$\theta = \delta - l_1$$

- Eliminating RHS: multiply
- Eliminating LHS: subtract

4 SCARA manipulator example

Break down the the inverse kinematics for the SCARA manipulator in Fig. 4 into simpler PK sub-problems.

$$\cancel{\left\| e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \left(e^{\hat{\xi}_3\theta_3} \cdot p - q \right) \right\| = \| gp - q \|}$$

$$\cancel{\| e^{\hat{\xi}_3\theta_3} p - q \| = \| gp - q \|}$$

No ~~q~~ that allows us to do this

$$\cancel{e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p = gp}$$

$$\cancel{e^{\hat{\xi}_1\theta_1} p = gp}$$

no p that allows us to do this

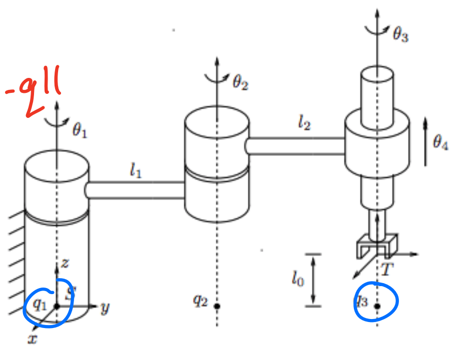


Figure 4: SCARA manipulator.

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} g_{st}(0) = g_d$$

Step 1: Solve for θ_4

- ξ_4 is the only joint that causes movement in z-direction
- We know our final z position

$$g_d = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

$$z_f = t_z$$

- Know initial $z = l_0$

$$\theta_4 = z_f - z_i$$
$$= t_z - l_0$$

✓ Solved for θ_4

Step 2: Solve for θ_2

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} = \underline{g_d\, g_{st}^{-1}(0)\, e^{-\hat{\xi}_4\theta_4}} =: g_1$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \underbrace{e^{\hat{\xi}_3\theta_3} q_3} = g_1 q_3$$

q_3 is on ξ_3

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdot q_3 = g_1 q_3$$

q₁ on ξ₁ → Magnitude → Rigid transform

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdot q_3 - q_1 = g_1 q_3 - q_1$$

q_1 on ξ_1

$$e^{\hat{\xi}_1\theta_1}\left(e^{\hat{\xi}_2\theta_2} q_3 - q_1\right) = g_1 q_3 - q_1$$

Magnitude

$$\left\| e^{\hat{\xi}_1\theta_1}\left(e^{\hat{\xi}_2\theta_2} q_3 - q_1\right)\right\| = \left\| g_1 q_3 - q_1 \right\|$$

Rigid transform

$$\left\| e^{\hat{\xi}_2\theta_2} q_3 - q_1 \right\| = \left\| g_1 q_3 - q_1 \right\|$$

$\boxed{\longrightarrow \text{Solve } \theta_2 \text{ w/ SP 3}}$

Step 3: Solve for θ_1

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \underbrace{e^{\hat{\xi}_3\theta_3} \cdot q_3} = g_1 \cdot q_3$$

$$e^{\hat{\xi}_1\theta_1}\left(\underbrace{e^{\hat{\xi}_2\theta_2} q_3}\right) = g_1 q_3$$

Know this value!
We know θ_2

$\rightarrow$ Solve for θ_1 w/ SP 1

Step 4: Solve θ_3

$$e^{\hat{\xi}_3\theta_3} = e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_d g_{st}^{-1}(0) e^{-\hat{\xi}_4\theta_4} =: g_2$$

Apply this to any pt. not on ξ_3

$$e^{\hat{\xi}_3\theta_3} \cdot q_2 = g_2 \cdot q_2$$

$$\boxed{\rightarrow \theta_3 \text{ w/ SP 1}}$$

Max # of sols: $1 \times 2 \times 1 \times 1 = \boxed{2}$

5 Elbow manipulator example

Break down the inverse kinematics for the elbow manipulator in Fig. 5 into simpler PK subproblems. Find the reachable and dexterous workspaces.

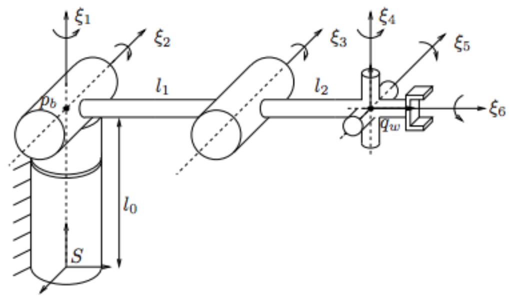


Figure 5: Elbow manipulator.