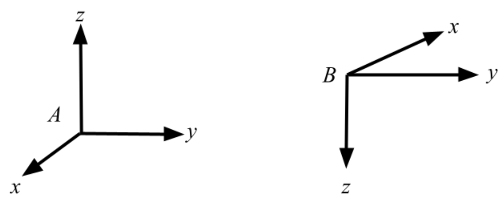
106B Background Assessment

Spring 2023

This short assessment is designed to test your background in some fundamental concepts required for 106B. You may refer to any material posted on the 106A website or any textbooks when completing this assessment, but may not use the open internet.

Problem 1: Rotation Matrices

Find the rotation matrix R_{ab} between the two orthonormal frames below:



Assume any intermediate rotations about the x, y, z axes are of angles that are integer multiples of $\frac{\pi}{2}$.

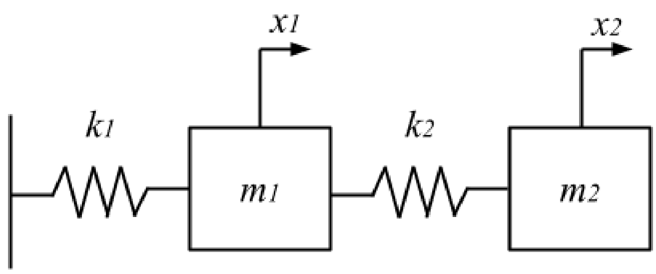
Problem 2: Matrix Exponential

1. If λ_i is an eigenvalue of $A \in \mathbb{R}^{n \times n}$, prove that e^{λ_i} is an eigenvalue of e^A.
2. Prove that if $\lambda_1, \lambda_2, ..., \lambda_n$ are the n potentially repeated eigenvalues of A, it is true that:

$$\det(e^A) = e^{\lambda_1 + \lambda_2 + ... + \lambda_n} \tag{1}$$

Problem 3: Dynamics

A system of two masses, m_1 and m_2 are attached by ideal springs to a rigid wall as follows:



The springs have constants k_1 and k_2 and the masses have positions x_1 and x_2. Assuming x_1 and x_2 are zero when the springs are unstretched, find the set of second order differential equations describing the motion of the masses. You may use either a Newtonian or Lagrangian approach. Assume that there are no frictional or gravitational forces.

Problem 4: Controls

Suppose a physical system is described by the following equations of motion:

$$m\ddot{x} = -mge_3 - \frac{1}{2}\rho C_d A||\dot{x}||\dot{x} + f \tag{2}$$

Where $x \in \mathbb{R}^3$, $e_3 = [0, 0, 1]^T$, m, g, ρ, C_d, A are constant scalars, and $f \in \mathbb{R}^3$ is an input to the system. If $x_d \in \mathbb{R}^3$ is the desired state of the system, find an expression for f such that the system error $e = x_d - x$ is described by the following differential equation:

$$\ddot{e} + k_d\dot{e} + k_p e = 0 \tag{3}$$

Where $k_p, k_d \in \mathbb{R}$ are arbitrary scalar constants. You may leave your expression for f in terms of any of the variables above and their derivatives.