Steering Nonholonomic Systems

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From Murray, Li, Sastry, Chapter 8
Sastry, Chapter 13, Nonlinear Systems

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Outline

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Nonholonomic Integrator

Consider the so-called nonholonomic integrator:

$$\begin{aligned} \dot{q}_1 &= \quad u_1 \\ \dot{q}_2 &= \quad u_2 \\ \dot{q}_3 &= \quad q_1 u_2 - q_2 u_1 \end{aligned}$$

This system has

$$g_1 = \begin{bmatrix} 1 \\ 0 \\ -q_2 \end{bmatrix} \quad g_2 = \begin{bmatrix} 0 \\ 1 \\ q_1 \end{bmatrix} \quad [g_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

This system is written by dropping the dt as

$$dq_3 = q_1 dq_2 - q_2 dq_1$$

Optimal Control

The optimal input minimizing the cost function

$$\int_0^1 ||u(t)||^2 dt$$

from an initial $q(0)$ to a final $q(1)$ was shown by Brockett to be sinusoidal. The frequency λ of the optimal inout is striking when $q_1(0) = q_1(1)$ and $q_2(0) = q_2(1)$ to be $2n\pi$ with $n = 0, \pm 1, \pm 2, \ldots$. The generalization of this system to $m > 2$ inputs is stated as a control system on $q \in \Re^m \times Y \in so(m)$ as

$$\begin{aligned} \dot{q} = &\quad u \\ \dot{Y} = &\quad qu^T - uq^T \end{aligned}$$

If q(0) = q(1) and $Y(1) \in so(m)$ is given then it can be shown that the optimal input is multiples of 2π, that is

$$\begin{array}{ll} 2\pi.2, \ldots, 2\pi\frac{m}{2} & \text{m even} \\ 2\pi, 2\pi.3, \ldots, 2\pi\frac{m-1}{2} & \text{m odd} \end{array}$$

Outline

Definition of a 1-chain form

The preceding discussion motivates some generalizations. First the non-holonomic integrator extended to dimension n

$$\begin{aligned}
\dot{q}_1 &= u_1 \\
\dot{q}_2 &= u_2 \\
\dot{q}_3 &= q_2 u_1 \\
\dot{q}_4 &= q_3 u_1 \\
&\vdots \\
\dot{q}_n &= q_{n-1} u_1
\end{aligned}$$

These are called **chained form** or **Goursat normal form systems**.

Controllability of a One Chain system

$$[g_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad [g_1, [\ldots, [g_1, \text{ k times, } g_2] \ldots]] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ (-1)^k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

By way of notation we define

$$ad_{g_1} g_2 = [g_1, g_2] \qquad ad_{g_1}^{k+1} = [g_1, ad_{g_1}^k g_2]$$

Thus

$$\text{Span}\{g_1, g_2, ad_{g_1}^k g_2, k = 1, \ldots, n-2\} = \Re^n$$

Chained Form Systems and Feedback Linearization

Consider the chained form system system just defined and set $u_1 = 1$. Then, $q_1 = t$ and we have

$$\begin{aligned} \dot{q}_2 = & \quad u_2 \\ \dot{q}_3 = & \quad q_2 \\ \dot{q}_4 = & \quad q_3 \\ & \vdots \\ \dot{q}_n = & \quad q_{n-1} \end{aligned}$$

This is precisely a linear system chain of integrators. More generally given a control system

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

when is it possible to transform this system into linear form with $u_1 = 1$, The answer is

$$\begin{aligned} &\{g_1, ad_{g_1}g_2, ad_{g_1}^2 g_2, \ldots, ad_{g_1}^{n-3}g_2\} && invol \\ &\{g_1, ad_{g_1}g_2, ad_{g_1}^2 g_2, \ldots, ad_{g_1}^{n-3}g_2, ad_{g_1}^{n-2}g_2\} && dim = n-1 \end{aligned}$$

Note that only $n-1$ variables are linearized, the n^{th} is just t.

Rectification using sinusoids

In analogy with the non-holonomic integrator first steer q_1, q_2. Then use $u_i(t) = a \sin 2\pi t$, $u_2(t) = b \cos 2\pi t$ to steer z_3.

$$\begin{aligned} q_1(t) = &\quad q_1(0) - \tfrac{a}{/} 2\pi(\ cos 2\pi t - 1) \\ q_2(t) = &\quad q_2(0) + \tfrac{b}{2\pi} \sin 2\pi t \\ q_3(t) = &\quad q_3(0) + \textstyle\int_0^t \tfrac{ab}{2\pi} \sin^2 2\pi t \end{aligned}$$

In one second $q_1(1) = q_1(0), q_2(1) = q_2(0)$. But because $\sin^2 2\pi t = \frac{1 - \cos 2.2\pi t}{2}$ it follows that

$$q_3(t) = q_3(0) + \frac{1}{2} \frac{ab}{2\pi} (t - \frac{\sin 2.2\pi t}{2.2\pi})$$

Because of the constant term (**rectification**) in the integrand after one second, we get an increase in q_3:

$$q_3(1) = q_3(0) + \frac{1}{2} \frac{ab}{2\pi}$$

Steering Chained Form Systems

Algorithm 3. Steering chained form systems

1. Steer q_1 and q_2 to their desired values.

2. For each q_{k+2}, $k \geq 1$, steer q_k to its final value using $u_1 = a \sin 2\pi t$, $u_2 = b \cos 2\pi kt$, where a and b satisfy

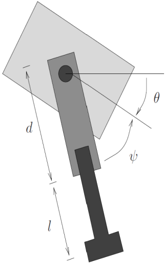
$$q_{k+2}(1) - q_{k+2}(0) = \left(\frac{a}{4\pi}\right)^k \frac{b}{k!}.$$

We use sinusoidal signals $u_1(t) = \sin 2\pi t$ and $u_2 = \cos 2\pi kt$ to steer q_k at $t = 1$ without changing the preceding $q_i, i < k$ for 1 second.

Steering Calculation

$$
\begin{aligned}
q_1 &= \frac{a}{2\pi}(1 - \cos 2\pi t), \\
q_2 &= \frac{b}{2\pi k} \sin 2\pi k t \\
q_3 &= \int \frac{ab}{2\pi k} \sin 2\pi k t \sin 2\pi t \, dt \\
&= \frac{1}{2} \frac{ab}{2\pi k} \left(\frac{\sin 2\pi (k-1) t}{2\pi (k-1)} - \frac{\sin 2\pi (k+1) t}{2\pi (k+1)} \right) \\
q_4 &= \frac{1}{2} \frac{a^2 b}{2\pi k \cdot 2\pi (k-1)} \int \sin 2\pi (k-1) t \cdot \sin 2\pi t \, dt + \cdots \\
&= \frac{1}{2^2} \frac{a^2 b}{2\pi k \cdot 2\pi (k-1) \cdot 2\pi (k-2)} \sin 2\pi (k-2) t + \cdots \\
&\vdots \\
q_{k+2} &= \int \frac{1}{2^{k-1}} \frac{a^k b}{2\pi k \cdot 2\pi (k-1) \cdots 2\pi} \sin^2 2\pi t \, dt + \cdots \\
&= \frac{1}{2^{k-1}} \frac{a^k b}{(2\pi)^k k!} \frac{t}{2} + \cdots
\end{aligned}
$$

Outline



With $q = (\psi, l, \theta)^T$, the control system of the hopper is

$$\dot{q} = \begin{bmatrix} 1 \\ 0 \\ -\frac{m(l+d)^2}{I+m(l+d)^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2$$

If you define $\alpha = \theta + \frac{md^2}{I+md^2}\psi$, we get for suitably defined f

$$\dot{\alpha} := f(l)u_1$$

Fourier Technique

Now choose

$$u_1 = a_1 \sin(2\pi t)$$
$$u_2 = a_2 \cos(2\pi t)$$

By Fourier series after integrating for l we get

$$f(l) = f(\frac{a_2}{2\pi} \sin 2\pi t) = \beta_1(a_2) \sin 2\pi t + \beta_2(a_2) \sin 4\pi t + \cdots$$

Using this in the equation for $\dot{\alpha}$ gives

$$\alpha(1) - \alpha(0) = \frac{1}{2} a_1 \beta_1$$

After 1 second, ψ, l are back to their initial values but α has changed by $\frac{1}{2} a_1 \beta_1(\alpha_2)$! By suitably choosing a_1, a_2 we can make this be π radians (a flip).!

Steering the kinematic car

The control systems for the car is

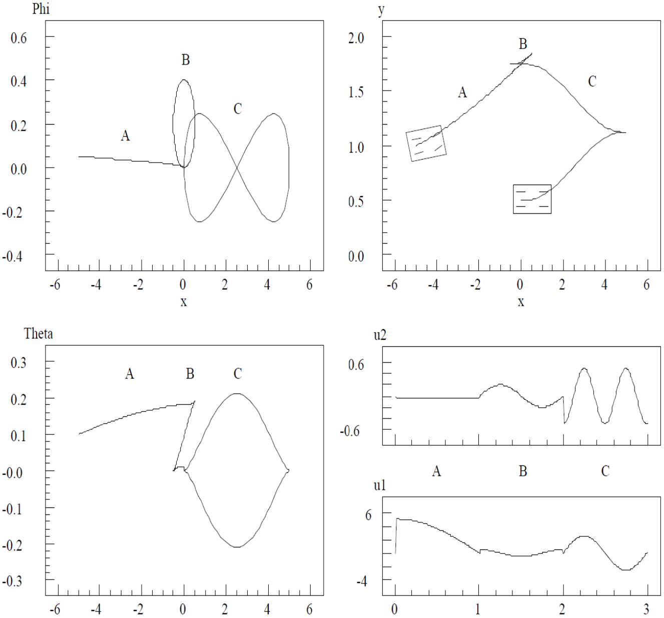
$$\begin{aligned}\dot{x} &= \cos\theta\, u_1\\ \dot{y} &= \sin\theta\, u_1\\ \dot{\theta} &= \frac{1}{l}\tan\phi\, u_1\\ \dot{\phi} &= u_2\end{aligned}$$

Change coordinates $z_1 = x, z_2 = \phi, z_3 = \sin\theta, z_4 = y$, inputs $v_1 = \cos\theta\, u_1, v_2 = u_2$ to get

$$\begin{aligned}\dot{z}_1 &= v_1\\ \dot{z}_2 &= v_2\\ \dot{z}_3 &= \frac{1}{l}\tan z_2 v_1\\ \dot{z}_4 &= \frac{z_2}{\sqrt{1-z_3^2}} v_2\end{aligned}$$

Now the linear terms in the last two equations match those of the one chain system and it can be steered using the algorithm with the Fourier series method and double frequency sinusoids

Parking a Car



Outline

Conversion to One Chained Form

Given a control system of the form in $\Re^n$ with two inputs

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

If (and only if) the folowing three distributions are regular (constant dimension) and involutive

$$\begin{aligned}
\Delta_0 &= \text{span}\{g_1, g_2, ad_{g_1} g_2, \ldots, ad_{g_1}^{n-2} g_2\} = \Re^n \\
\Delta_1 &= \text{span}\{g_2, ad_{g_1} g_2, \ldots, ad_{g_1}^{n-2} g_2\} \\
\Delta_2 &= \text{span}\{g_2, ad_{g_1} g_2, \ldots, ad_{g_1}^{n-3} g_2\}
\end{aligned}$$

then there exists a choice of coordinates $z_1(q), \ldots, z_n(q)$ and inputs $v = \alpha(q) + \beta(q)u$ auch that

$$\begin{aligned}
\dot{z}_1 &= \quad v_1 \\
\dot{z}_2 &= \quad v_2 \\
\dot{z}_3 &= \quad z_2 v_1 \\
&\vdots \\
\dot{z}_n &= \quad z_{n-1} v_1
\end{aligned}$$

Feedback Linearization and Chained Forms

Amazingly chained form systems are closely related to feedback linearization. This is how you see this. If you set $u_1 = 1$ we get a single input control system

$$\dot{x} = g_1(x) + g_2(x)u_2$$

The conditions for full state feedback linearization of this system are

$$\begin{aligned}\Delta_0 &= \text{span}\{g_1, g_2, ad_{g_1}g_2, \ldots, ad_{g_1}^{n-2}g_2\} = \Re^n \\ \Delta_1 &= \text{span}\{g_2, ad_{g_1}g_2, \ldots, ad_{g_1}^{n-2}g_2\}\end{aligned}$$

Indeed if we define $\Delta_1 dh$ and define $w_1 = h(x), z_2 = L_{g_1}h(x), \ldots, w_n = L_{g_1}^{n-1}h(x)$ we can check that when we apply this to the original control system without u_1 frozen at 1.

$$\begin{aligned}\dot{w}_1 = &\quad w_2 u_1 \\ \dot{w}_2 = &\quad w_3 u_1 \\ \dot{w}_3 = &\quad w_4 u_1\end{aligned}$$

Feedback Linearization and Chained Forms

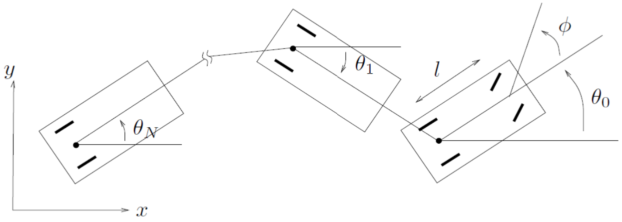
if you set $v_2 = u_1$ and $v_1 = L_{g_1} L_{g_2}^{n-1} u_1 - L_{g_2}^{n} h u_2$, we get

$$\begin{aligned}
\dot{w}_1 = &\quad w_2 v_2 \\
\dot{w}_2 = &\quad w_3 v_2 \\
\dot{w}_3 = &\quad w_4 v_2 \\
&\vdots \\
\dot{w}_n = &\quad v_1
\end{aligned}$$

This is exactly the chained form with the numbering of the states inverted! Thus, if you set $z_i = w_{n-i}$, we get exactly the chained form solution.

Conversion to One Chained form

Amazingly, all of the examples: so far: cars, unicycles, pennies with two inputs can be converted into a single chain form eactlly and steered as above !!
Amazingly so is the the **Car with N Trailers**



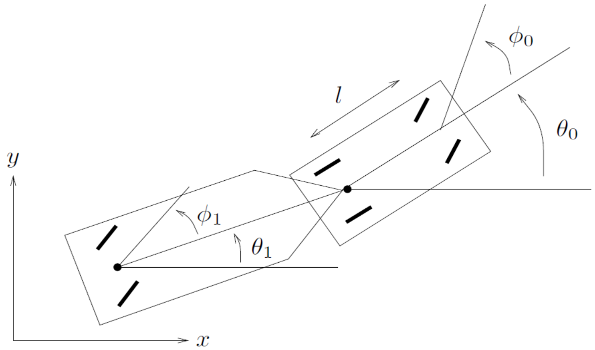
The state space $q = (x, y, \phi, \theta_0, \ldots, \theta_N)^T \in \Re^{N+4}$. The coordinates z_1, z_2 in the chained form are the x, y coordinates of the last trailer wheel base!!

Multi Chained Forms

When there are $m \geqslant 3$ inputs for

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2 + \cdots + g_m(q)u_m$$

there are analogous necessary and sufficient conditions for the conversion into $m-1$ chains. For example, the **firetruck**. The driver in front has two inputs: drive and steer and the one at the back of the ladder can steer. This can be converted into a two chain system.



Conversion to Multi-Chained Form

In analogy to the two input case, we may ask if it is possible to find one variable with new input $v_1 = 1$ and have $(m-1)$ other chains of linear inputs. The general solution to this requires some work but you can already see thatif $u_1 = v_1 = 1$, then we can treat g_1 like the drift vector field and we would have

$$\dot{q} = g_1(q) + g_2(q)u_2 + \cdots + g_m(q)u_m$$

and the involutivity conditions would be for the filtration

$$\begin{aligned}
&\Delta_1 = \{g_2, \ldots, g_m\} \\
&\Delta_2 = \{g_2, \ldots g_m, ad_{g_1} g_2, \ldots, ad_{g_1} g_2\} \\
&\Delta_3 = \{g_2, \ldots, g_m, ad_{g_1} g_2, \ldots, ad_{g_1} g_m, ad^2_{g_1} g_2, \ldots, ad^2_{g_1} g_m\} \\
&\vdots
\end{aligned}$$

This is generalizable as discussed in Chapter 13 if we do not want to legislate that $u_1 = 1$.

Thank you for your attention. Questions?

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