C106B Discussion 5: Kinematic Constraints

1 Introduction

Today, we'll talk about:

1. Pfaffian Constraints
2. Equivalent Control Systems
3. Lie Brackets & Controllability

2 Pfaffian Constraints

When performing path planning tasks in robotics, it's essential to have an understanding of how our system moves, as we *always* want to generate paths that are feasible for our system to follow! It's therefore important for us to understand the *kinematic constraints* on a system - the constraints that impact the possible positions and velocities of the system.

Let's consider a physical system with generalized coordinates $q_1, q_2, ..., q_n$. We know that using Lagrangian mechanics, we can find the dynamics of the system by computing n differential equations:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = F_i \tag{1}$$

A *kinematic constraint* imposes restrictions on the generalized coordinates and their velocities. A *Pfaffian constraint* is a constraint of the form:

$$\omega_i(q)\dot{q} = 0 \tag{2}$$

Where q is a vector of the system's generalized coordinates. A Pfaffian constraint on the velocities of q_i is said to be *integrable* if it is equivalent to a constraint on the *positions* of q_i:

$$\omega_i(q)\dot{q} = 0 \Longleftrightarrow h_i(q) = 0 \tag{3}$$

If a set of k Pfaffian constraints ω_i are all integrable, the set of constraints is said to be *holonomic.* If a subset of the constraints are integrable, then the constraints are said to be *partially nonholonomic.* If no constraints are integrable, the set is *completely nonholonomic.*

Problem 1: A uniform, rigid pendulum of length $2L$ swings about a pivot point. The angle of the pendulum to the vertical is θ and the position of the center of mass is (x, y). Write the constraints on the values of x, y subject to the pendulum's motion. Are these constraints holonomic?

3 Equivalent Control Systems

Suppose we have k independent, nonholonomic Pfaffian constraints $\omega_1(q)\dot{q} = 0, ..., \omega_k(q)\dot{q}$, where $q \in \mathbb{R}^n$ is a vector of generalized coordinates. We can write these constraints in matrix form $A(q)\dot{q} = 0$ as:

$$\begin{bmatrix} - & \omega_1(q) & - \\ & \vdots & \\ - & \omega_k(q) & - \end{bmatrix} \dot{q} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4}$$

We know that the columns of $A(q)$ above are linearly independent, and that all allowable trajectories $\dot{q}$ must be in the null space of $A(q)$. Since $q \in \mathbb{R}^n$, and the matrix is in $\mathbb{R}^{k \times n}$ with k independent constaints, the null space must be $n - k = m$ dimensional.

Therefore, there exist $g_1(q), ..., g_m(q)$ that span the basis of this null space such that:

$$\dot{q} = u_1 g_1(q) + ... + u_m g_m(q) \tag{5}$$

Are all allowable trajectories, where $u_i \in \mathbb{R}$ are scalars. Since we can arbitrarily control u_i, we have found an *equivalent control system* for our dynamics just using the kinematic constraints. This equivalent control system is a simpler model that expresses what it means for a trajectory to be allowable, and we can use it to control our system's generalized coordinates. Note that each $g_i(q) \in \mathbb{R}^n$ is called a *vector field*, as it maps a vector to a vector.

Problem 2: The Raibert hopper, which has generalized coordinates $q = [\phi, l, \theta]^T$ has the following nonholonomic constraint on its dynamics.

$$I\dot{\theta} + m(l + d)^2(\dot{\theta} + \dot{\phi}) = 0 \tag{6}$$

Rewrite this constraint in the form $A(q)\dot{q} = 0$, and find a basis for the null space of $A(q)$.

4 Lie Brackets & Controllability

How can we use a nonholonomic constraint $\dot{q} = A(q)q, q \in \mathbb{R}^n$ to design feedback controllers and path planners for our system? Let's imagine that we want to drive our system to the position $q_d \in \mathbb{R}^n$? Under our kinematic constraints, is it actually possible to steer our system to q_d? To answer this question, we'll need a few tools from the field of differential geometry.

Definition 1 ***Lie Bracket***
The Lie bracket of two vector fields $f(q), g(q)$ is defined:

$$[f, g](q) = \frac{\partial g}{\partial q} f(q) - \frac{\partial f}{\partial q} g(q) \tag{7}$$

The Lie bracket measures whether flows of equal time along f and g commute.

Definition 2 ***Lie Algebra***
The Lie algebra of a set of vector fields $\{g_1, g_2\}$, denoted $\mathcal{L}(g_1, g_2)$, is the span of all linear combinations of g_1, g_2, their Lie brackets, and higher order Lie brackets:

$$g_1, g_2, [g_1, g_2], [g_1, [g_1, g_2]], [g_2, [g_1, g_2]], ... \tag{8}$$

For a set of m vector fields, $g_1, ..., g_m$, the Lie algebra $\mathcal{L}(g_1, ..., g_m)$ is similarly defined by taking the span of the vector fields and their Lie brackets with each other.

Here's the basic concept of our big idea for this section: if the vector fields $g_1(q), ..., g_m(q)$ from our equivalent control system have *nonzero* Lie brackets, we might be able to form a basis of vector fields we may travel along to reach any location.

Theorem 1 ***Small Time Local Controllability***
A system is small time locally controllable at a point q_0 if the set of states the system can reach in finite time starting from q_0 forms a ball around q_0. If the dimension of $\mathcal{L}(g_1, ..., g_m)$ is equal to the dimension of q, and the positive span of the vector $[u_1, ..., u_m]$ is $\mathbb{R}^m$, then the system:

$$\dot{q} = g_1(q)u_1 + ... + g_m(q)u_m \tag{9}$$

Is small time locally controllable.

Problem 3: Imagine we have a vector of generalized coordinates $q = [x, y, z]^T$. These coordinates have a kinematic constraint which may be represented by the control system:

$$\dot{q} = \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2 = g_1(q)u_1 + g_2(q)u_2 \tag{10}$$

Where u_1 and u_2 can have any values in $\mathbb{R}$. Find the Lie bracket $[g_1, g_2]$, conclude the Lie algebra $\mathcal{L}(g_1, g_2)$ has dimension 3, and show that the system is small time locally controllable.