

# C106B Discussion 6: Probability and CV

## 1 Introduction

In anticipation for the upcoming unit on environment feedback, localization, and mapping, today we'll talk about:

1. Linearity of Expectation
2. Multivariate Random Variables
3. Hidden Markov Models
4. Low-Level Computer Vision Review

$$Y \sim \text{Uniform}(0, 1) \\ \mathbb{E}[Y] = 0.5$$

Coin flip

$$\text{R.V. } X \rightarrow \begin{array}{ll} \text{Heads} & \text{w.p. } 0.5 \\ \text{Tails} & \text{w.p. } 0.5 \end{array}$$

## 2 Linearity of Expectation

*Random variables* are commonly used to perform probabilistic calculations. They take on values corresponding to some distribution. The average value of some random variable  $X$  is known as the *expectation* of  $X$ .

Linearity of expectation allows us to compute the average value of a combination of multiple random variables. The theorem states:

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\mathbb{E}[cX] = c\mathbb{E}[X] \rightarrow c \text{ is a constant}$$

$$\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$$

**Problem 1:** Suppose I toss all 10 (unique) pairs of my socks in the washing machine, but when I collect them from my dryer, I only have 16 socks remaining. In expectation, how many pairs can I expect to see?

$$X_i = \begin{cases} 0 & \text{if dryer eats at least 1 sock in pair } i \\ 1 & \text{if both socks survive} \end{cases}$$

$$P[X_i=1] \rightarrow \text{chance that this pair comes out} \\ = \underbrace{\frac{4}{20}}_{\text{prob 1st sock survives}} \times \underbrace{\frac{16}{19}}_{\text{chance 2nd sock lives}} = \frac{12}{19} = \mathbb{E}[X_i]$$

$$\mathbb{E}[\text{Total pairs}] = \# \text{ of pairs} \times \mathbb{E}[\text{any given pair}] \\ = 10 \times \frac{12}{19} \approx \boxed{6.32 \text{ pairs}}$$

### 3 Multivariate Random Variables

A *multivariate random variable*, also known as a *random vector*, can be thought of as a group of random variables that are associated with one another in a single mathematical system. For example, a robot's predicted (x, y, z) location might be the output of some probabilistic function of observed environment variables, and the coordinates are grouped together to represent the state.

#### 3.1 Mean, Covariance, and Cross-Covariance

The CDF takes a vector as input with the number of entries corresponding to the number of random variables:

$$F_X(x) = P(X_1 \leq x_1, \dots, X_n < x_n)$$

As a result, the mean, or **expected value**, of a multivariate random variable is a **vector** as well:

$$\mathbb{E}[X] = (\mathbb{E}[X_1], \dots, \mathbb{E}[X_n])^T$$

The *variance* of a single random variable equals the **average distance from the mean** of the values that it can take. It can be computed in two different ways:

$$\text{Var}(X) = \sigma^2(X) = \frac{\sum (x_i - \bar{x})^2}{N} = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

The **covariance** of multiple random variables **represents the extent to which they correspond in values**. If they both increase and decrease together, this value will be positive; whereas if they move in opposite directions, the value will be negative. Independent random variables will have 0 covariance (although the converse does not hold true).

$$\text{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

A single multivariate random variable will have an associated covariance matrix to represent pairwise covariance. Covariance matrices are symmetric positive semidefinite. A cross-covariance matrix can be calculated to represent the covariance between two different multivariate random variables. The element in the i, j position represents the covariance between the i-th value in the first vector and the j-th value in the second.

$$\text{cov}(\mathbf{X}, \mathbf{Y}) = \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{Y} - \mathbb{E}[\mathbf{Y}])^T] = \mathbb{E}[\mathbf{X}\mathbf{Y}^T] - \mathbb{E}[\mathbf{X}]\mathbb{E}[\mathbf{Y}]^T$$

**Problem 2:** Interpret the following ~~matrix~~ covariance matrix. Which of the values is/are invalid?

$$\begin{bmatrix} 3 & 5 & 0 \\ 5 & 12 & -7 \\ 6 & -7 & -4 \end{bmatrix}$$

→ variance is negative  
 Should be avg squared to mean distance  
 Not symmetric  
 $\text{Cov}(X_i, X_j) = \text{Cov}(X_j, X_i)$

Covariance:  $\text{Cov}(X, X)$   
 Cross-covariance:  $\text{Cov}(X, Y)$

$$\begin{bmatrix} X_1 - \mathbb{E}[X_1] \\ \vdots \\ X_N - \mathbb{E}[X_N] \end{bmatrix} \begin{bmatrix} Y_1 - \mathbb{E}[Y_1] & \dots & Y_N - \mathbb{E}[Y_N] \end{bmatrix}$$

$$= \begin{bmatrix} (X_1 - \mathbb{E}[X_1])(Y_1 - \mathbb{E}[Y_1]) & & \\ & \dots & \\ & & \dots \end{bmatrix}$$

### 3.2 Multivariate Gaussians

A multivariate Gaussian random variable, or normal distribution, follows the following distribution:

$$p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{n}{2}}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)\right) = c$$

The sum of Gaussians is a Gaussian:

$$X \sim N(\mu_X, \sigma_X^2)$$

$$Y \sim N(\mu_Y, \sigma_Y^2)$$

$$Z = X + Y$$

$$Z \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

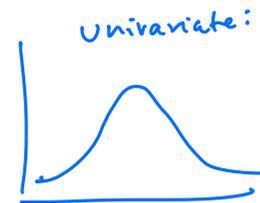
The PDF of a multivariate Gaussian with a diagonal covariance matrix will be the same as that of  $n$  independent Gaussians (uncorrelated implies independence).

**Problem 3:** The isocontours of a function  $f: \mathbf{R}^2 \rightarrow \mathbf{R}$  are of the form  $x \in \mathbf{R}^2 : f(x) = c$ . Find the isocontours of a multivariate Gaussian, both with and without a diagonal covariance matrix. What kind of intuition does this give you about Gaussians?



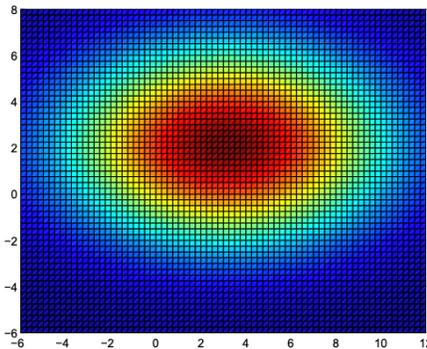
$$1 = \left(\frac{x_1 - \mu_1}{r_1}\right)^2 + \left(\frac{x_2 - \mu_2}{r_2}\right)^2$$

functions of  $c$



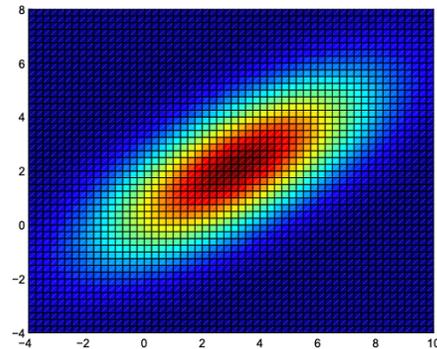
$$\Sigma = \begin{bmatrix} 25 & 0 \\ 0 & 9 \end{bmatrix}$$

$$\mu = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



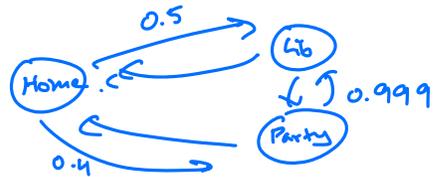
$$\Sigma = \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix}$$

$$\mu = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



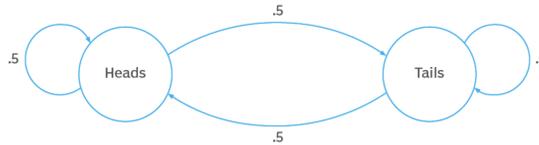
↳ singular peak  
Distributed elliptically around that peak

## 4 Hidden Markov Models



A *Markov Chain* possesses the Markov property - the next state depends only on the current state and is entirely independent of the past. For example, a coin toss can be expressed as a Markov Model:

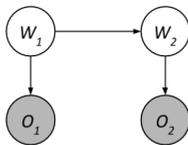
**Markov chain coin toss**



A *Hidden Markov Model* is often used in systems where data is continuously fed in over time. HMMs assume that states themselves have the Markov property. However, they are unknown - *observations* are used to form a probabilistic distribution of your current state. The observations could be sensor readings, for example, and the state might be the (unknown) location of your robot in the space. This is quite useful for localization!

↳ Use observations to determine where in state we could be

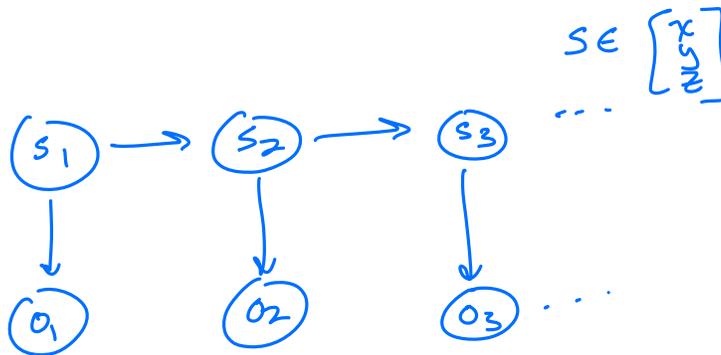
**Problem 4:** Suppose we observe  $O_1 = a$  and  $O_2 = b$ . Compute the probability distribution  $P(W_2|O_1 = a, O_2 = b)$ . [Source: CS 188 Fa22 Discussion 9]



| $W_1$ | $P(W_1)$ |
|-------|----------|
| 0     | 0.3      |
| 1     | 0.7      |

| $W_t$ | $W_{t+1}$ | $P(W_{t+1} W_t)$ |
|-------|-----------|------------------|
| 0     | 0         | 0.4              |
| 0     | 1         | 0.6              |
| 1     | 0         | 0.8              |
| 1     | 1         | 0.2              |

| $W_t$ | $O_t$ | $P(O_t W_t)$ |
|-------|-------|--------------|
| 0     | a     | 0.9          |
| 0     | b     | 0.1          |
| 1     | a     | 0.5          |
| 1     | b     | 0.5          |



$O_t \in$  sensor reading

①  $P(W_1, O_1 = a) \rightarrow$  probability distribution of 1<sup>st</sup> state & observation

$$= P(W_1 = w_1) \cdot P(O_1 = a | W_1 = w_1)$$

$$P(W_1 = 0; O_1 = a) = (0.3)(0.9) = 0.27$$

$$P(W_1 = 1, O_1 = a) = (0.7)(0.5) = 0.35$$

$$\textcircled{2} P(w_2, o_1 = a) = \sum_{w_1} \underbrace{P(w_1, o_1 = a)}_{\text{Computed in pt. a}} \cdot \underbrace{P(w_2 | w_1)}_{\text{Markov Model}}$$

$$P(w_2 = 0, o_1 = a) = (0.27)(0.4) + (0.35)(0.8) = 0.388$$

$$P(w_2 = 1, o_1 = a) = (0.27)(0.6) + (0.35)(0.2) = 0.232$$

$\textcircled{3}$  Update distribution with our new info about Observation 2

$$P(w_2, o_1 = a, o_2 = b)$$

$$P(w_2 = 0, o_1 = a, o_2 = b) = \underbrace{P(w_2 = 0, o_1 = a)}_{\text{Computed above}} \cdot \underbrace{P(o_2 = b | w_2 = 0)}_{\text{given}}$$

$$= 0.0388$$

$$P(w_2 = 1, o_1 = a, o_2 = b) = P(w_2 = 1, o_1 = a) \cdot P(o_2 = b | w_2 = 1)$$

$$= 0.116$$

$\textcircled{4}$  Normalize

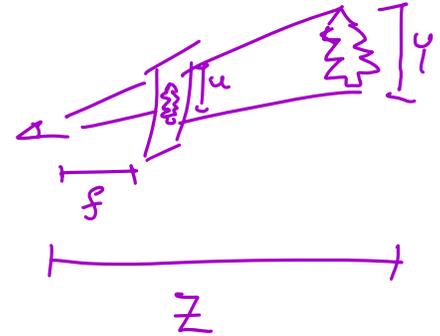
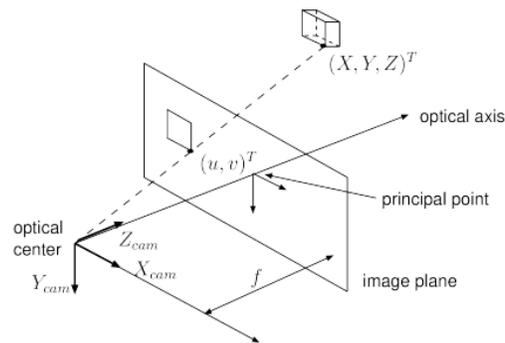
$$P(w_2 = 0 | o_1 = a, o_2 = b) = \frac{0.0388}{0.0388 + 0.116} \approx 0.25$$

$$P(w_2 = 1 | o_1 = a, o_2 = b) = \frac{0.116}{0.0388 + 0.116} \approx 0.75$$

\* Found a distribution of where we could be @  $t=2$  based on sensor readings

## 5 Computer Vision

### 5.1 The Pinhole Camera Model

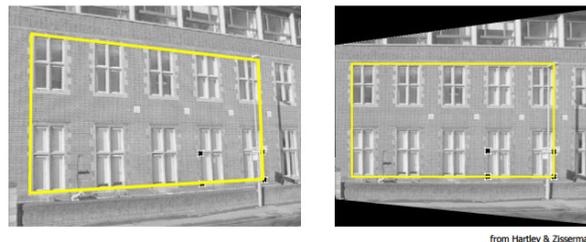


**Problem 5:** Using the image above, find the relationship between the 3D point  $(X, Y, Z)^T$  to its corresponding 2D projection  $(u, v)$  onto the imaging plane (assume the focal length is 1).

similar triangles  $\frac{u}{f} = \frac{y}{Z} \quad u = f \left( \frac{y}{Z} \right)$

### 5.2 Homography

The pinhole camera model has a particular center of projection, the point from which we view the world. The image plane is offset a certain focal length in some direction from that point. If we rotate the camera without moving it around, we maintain the same center of projection - we're just looking a different way! This is known as a *homography transformation* and can be thought of as a series of unprojection, rotation, and then reprojection. It's quite a useful function to have, especially when your robot is moving around. The homography transformation can be used to straighten images - if we've taken a picture with the camera pointed sideways, we can rotate it so that it looks as though the camera is pointed straight!



**Problem 6:** Let  $p$  correspond to a point on one image and let  $p'$  correspond to the same point in the scene, but projected onto another image. Write a general equation for how a homography matrix  $H$  maps points from one image to another. How would  $H$  be restricted if it must describe an affine transformation?

$$p' = Hp$$

↳ bottom row is  $[0 \ 0 \ 1]$

**Problem 7:** How can you compute a homography matrix with real-world points?

- Perform feature mapping
  - Identify matching pts. in each picture
  - Perform least squares