Lecture 18

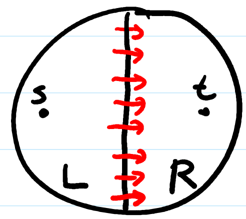
Monday, February 19, 2024 5:34 PM

LINEAR PROGRAMS

Minimal Cut:

Def.: Given

- a directed graph $G=(V,E)$
- capacities $c_e \geq 0$, $e \in E$
- a source $s \in V$, a sink $t \in V$



an $\boxed{\text{s-t cut}}$ is a partition $V = L \cup R$ s.th. $s \in L$, $t \in R$

$$\text{capacity}(L,R) = \sum_{\substack{uv \in E \\ u \in L, v \in R}} c_{uv}$$

$$\boxed{\text{Min Cut}} = \min_{(L,R)} \text{capacity}(L,R)$$

Maximum Flow Problem

Maximize $\text{size}(f) := \sum_{v: sv \in E} f_{sv}$ (flow s to t)

s.th. f is a flow, i.e.,

$$0 \leq f_e \leq c_e \qquad \text{for all } e \in E$$

$$\sum_{uv \in E} f_{uv} = \sum_{vw \in E} f_{vw} \qquad \text{for all } v \neq s,t$$

$$\boxed{\text{Max Flow}} = \max_{f \text{ is a flow}} \text{size}(f) \quad \left(\text{max. flow from } s \text{ to } t\right)$$

Thm: If the capacities are integers

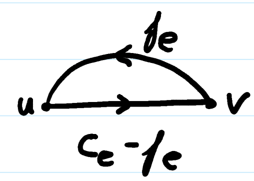
$$\boxed{\text{Min Cut} = \text{Max Flow}} = \text{size}(f)$$

where f is the output of the Ford-Fulkerson algor.

Residual Graph

Def: Given a graph G and a flow f on G, the residual graph G_f is obtained as follows:

For each edge $e = uv \in E$,



- the edge uv has capacity $c_e - f_e$
- we create back-edge vu with capacity f_e

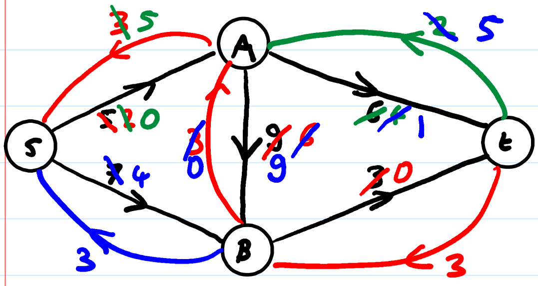
Ford Fulkerson Algorithm

1) Find path P from s to t which is not yet saturated in the residual graph G_f
2) Add flow $\max\limits_{e \in P} c_e(G_f)$ along P
3) Update the capities in the residual graph
4) Repeat until all paths P from s to t are saturated

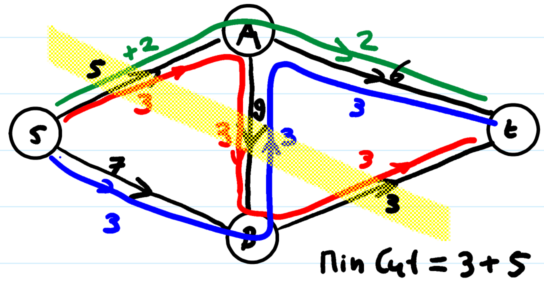
Rem: If the capacities are integers, Ford-Fulkerson assigns integer flows to all edges and terminates using at most $U = \text{Max Flow}$ "augmenting" paths P

Runtime: $\underbrace{\#\text{ of augmenting path}}_{\leq U = \text{max-flow}} \times \underbrace{\text{time to find paths}}_{O(n+m)}$ depth first search

Example:



Residual Graph G_f

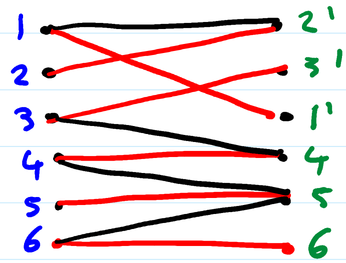


Size(Flow) = $2+3+3 = 8$

Bipartite Perfect Matching

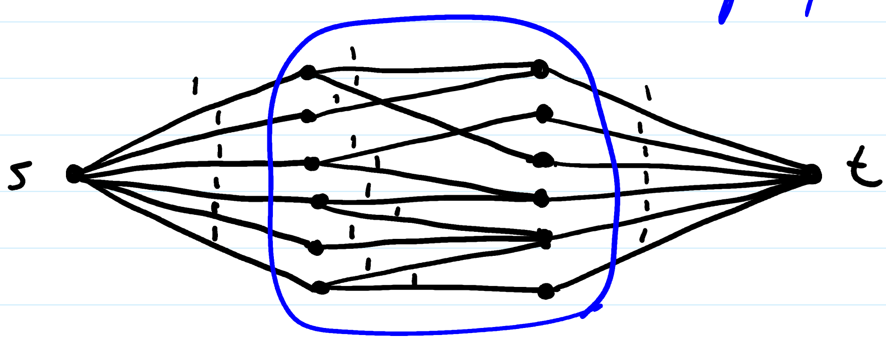
Input: Bipartite graph $G=(L,R,E)$ $|L|=|R|=n$

Output: A perfect matching M from L to R



$M \subseteq E$

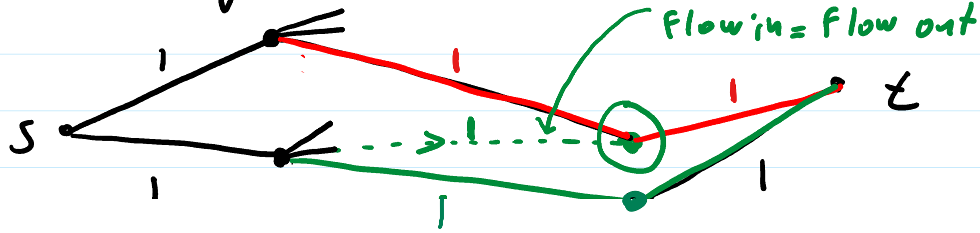
Solution via Max-Flow on new graph G'



G'

$\exists$ perfect matching $\Rightarrow$ $\underbrace{\text{we can send flow } n \text{ from } s \text{ to } t}_{*}$

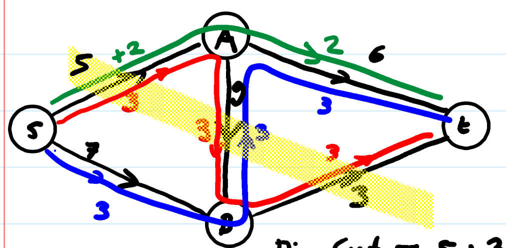
$* \Rightarrow \exists$ integer flow n



$\Rightarrow \exists$ perfect matching

$\Rightarrow$ We can solve the problem by running Ford-Fulkerson on G'

LP Duality



min Cut = 5+3=8

Last time, we used Max Flow $\leq$ Min Cut to prove that size(f) = 8 is optimal

Works much more general!

Example:

$$\max\ 5x_1 + 4x_2$$

s.th. $2x_1 + x_2 \leq 100$

$x_1 \leq 30$

$x_2 \leq 60$

$x_1, x_2 \geq 0$

Solution: $x_1 = 20$, $x_2 = 60$
value $= 340$

$$\max\ 5x_1 + 4x_2$$

s.th. $2x_1 + x_2 \leq 100$

$(x_1 \leq 30) \cdot 5$

$(x_2 \leq 60) \cdot 4$

$$5x_1 + 4x_2 \leq \underbrace{150 + 240}_{390}$$

$$\max\ 5x_1 + 4x_2$$

s.th. $(2x_1 + x_2 \leq 100) \cdot 3$

$x_1 \leq 30$

$(x_2 \leq 60) \cdot 1$

$$6x_1 + 4x_2 \leq 360$$

$$\max\ 5x_1 + 4x_2$$

s.th. $(2x_1 + x_2 \leq 100) \cdot \frac{5}{2}$

$x_1 \leq 30$

$(x_2 \leq 60) \cdot \frac{3}{2}$

$$5x_1 + 4x_2 \leq \underbrace{250 + 90}_{340}$$

How did we get these magic numbers $5/2$, $3/2$

Primal LP

$$\begin{cases} \max & 5x_1 + 4x_2 \\ \text{s.th.} & (2x_1 + x_2 \le 100)\ y_1 \\ & (x_1 \le 30)\ y_2 \\ x_1, x_2 \geq 0 & (x_2 \le 60)\ y_3 \end{cases}$$

$$\Rightarrow (2y_1 + y_2)x_1 + (y_1 + y_3)x_2 \le 100y_1 + 30y_2 + 60y_3$$

as long as $y_1, y_2, y_3 \geq 0$

Best upper bd. on $5x_1 + 4x_2$

Dual LP

$$\begin{cases} \min & 100y_1 + 30y_2 + 60y_3 \\ \text{s.th.} & y_1, y_2, y_3 \geq 0 \\ & 2y_1 + y_2 \geq 5 \\ & y_1 + y_3 \geq 4 \end{cases}$$

By Construction: $\boxed{5x_1 + 4x_2 \le 100y_1 + 30y_2 + 60y_3}$

$\Rightarrow$ Primal LP OPT $\le$ Dual LP OPT

General Case:

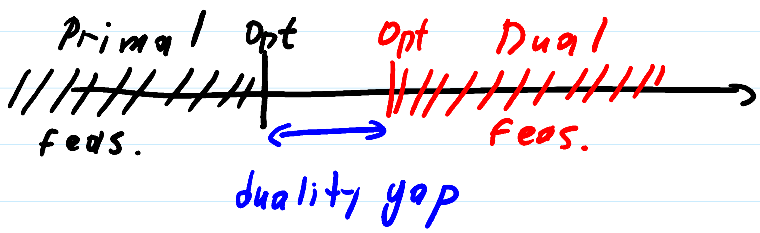
Primal LP	Dual LP
$\max\ c^T x$	$\min\ b^T y$
s.th. $x \geq 0$	s.th. $y \geq 0$
$Ax \le b$	$A^T y \geq c$

Thm: $\forall$ feasable sol. x of primal
$\forall$ —— " —— y of dual

$$\text{value}(x) \le \text{value}(y)$$

Pf: $c^T x \le y^T A x \le y^T b = \sum_i y_i b_i = b^T y$

Cor: Opt Primal $\le$ Opt Dual (weak duality)



Thm[Strong Duality]: If the primal has a bounded optimum $\Rightarrow$ Opt Primal = Opt Dual

Ex.: Max Flow = Min Cut

More General Duality

Primal:

$\max c^T x$

s.t. $(Ax)_i \le b_i \quad i \in I$

$(Ax)_i = b_i \quad i \notin I$

$x_j \ge 0 \quad j \in P$

Dual

$\min b^T y$

s.t. $(A^T y)_j \ge c_j \quad j \in P$

$(A^T y)_j = c_j \quad j \notin P$

$y_i \ge 0 \quad i \in I$

Proof of weak duality [not exam material]:

First constraint + $y_i \geq 0$ for $i \in I$ $\Rightarrow$ $\sum_j y_i A_{ij} x_j \leq b_i y_i$ if $i \in I$

Second — " — $+ y_i \in \mathbb{R}$ for $i \notin I$ $\Rightarrow$ $\sum_j y_i A_{ij} x_j = b_i y_i$ if $i \notin I$.

$$\sum_j (A^T y)_j x_j = \sum_{i,j} y_i A_{ij} x_j \leq \sum_i b_i y_i = b^T y \quad (*)$$

We want the LHS to be an upper bd on $c^T x = \sum_j c_j x_j$. This follows from the dual constraints + $x_j \geq 0$ for $j \in P$

$x_j \geq 0$ and $(A^T y)_j \geq c_j \Rightarrow (A^T y)_j x_j \geq c_j x_j$ if $j \in P$

$x_j \in \mathbb{R}$ and $(A^T y)_j = c_j \Rightarrow (A^T y)_j x_j = c_j x_j$ if $j \notin P$

$$\sum_j (A^T y)_j x_j \geq \sum_j c_j x_j = c^T x$$

Together with $(*)$, we got the weak duality claim

$c^T x \leq b^T y$ whenever x, y are feasible ∎

Two Player - Zero Sum Games

Input: Payoff Matrix M

	rock	pape	siss.
rock	0	-1	1
paper	1	0	-1
sciss.	-1	1	0

Row Player: picks row r } Payoff $\{ M[r,c]$

Col. Player: picks col c } Payoff $\{ -M[r,c]$

2 types of strategies

"Pure strategy": a single row/column

e.g. row always plays rock (beaten by paper)

"Mixed strategy": probability distribution over pure strategies, e.g.

$Pr[\text{Rock}] = \frac{1}{3}$, $Pr[\text{Paper}] = \frac{1}{3}$ $Pr[\text{Siss.}] = \frac{1}{3}$

Note: Average Payoff is 0, no matter what row plays.

Holds for general zero sum game!

Who goes first?

Game 1:

	1	2
1	3	-1
2	-2	1
	q_1	q_2

Turn Order:

1. Row player announces mixed strategy $p = (p_1, p_2)$

   $p_1 = Pr[\text{Row } 1]$
   $p_2 = Pr[\text{Row } 2]$

2. Col player responds w/ mixed strategy $q = (q_1, q_2)$

Def: Row player's average score: $\text{Score}(p,q) = 3p_1q_1 - 1p_1q_2 - 2p_2q_1 + 1p_2q_2$

Col player's best response

$$\underset{\text{mixed strat. } q}{\text{minimize}} \ \text{Score}(p, q)$$

$$= \underset{\text{pure strat.}}{\min} \{ \underbrace{3p_1 - 2p_2}_{\text{row } 1}, \underbrace{-p_1 + p_2}_{\text{row } 2} \}$$

Row players best strategy

$$\underset{\text{mixed strat. } p}{\text{maximize}} \ \min\{3p_1 - 2p_2, -p_1 + p_2\}$$

Claim: This is a LP

Pf: max X

s.th. $X \le 3p_1 - 2p_2$

$X \le -p_1 + p_2$

$p_1, p_2 \ge 0,\ p_1 + p_2 = 1$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}\ X = \min\{3p_1 - 2p_2, -p_1 + p_1\}$

Game 2 [col. player goes first]

	1	2
1	3	-1
2	-2	1
	q_1	q_2

Given q, payoff row 1 : $3q_1 - q_2$

―― ‖ ―― 2 : $-2q_1 + q_2$

Row players best resp. $\max\{3q_1 - q_2, -2q_1 + q_2\}$

Col players best strat. $\min_{\text{mixed strat. } q} \max\{3q_1 - q_2, -2q_1 + q_2\}$

LP 2: min Y

s.th. $Y \ge 3q_1 - q_2$

$Y \ge -2q_1 + q_2$

$q_1, q_2 \ge 0,\ q_1 + q_2 = 1$

Game 1:

Row player first
Col. ―‖― 2^{nd}

	1	2
1	3	-1
2	-2	1

Game 2:

Col. player first
Row ―‖― 2^{nd}

$$\max_p \min_q \text{Score}(p,q) \ \cancel{\leq}_{\text{weak duality}} \overset{\text{strong duality}}{=} \min_q \max_p \text{Score}(p,q)$$

General Zero-Sum, 2 Player Game

$$\max_p \min_q \sum p_r M[r,c] q_c = \min_q \max_p \sum p_r M[r,c] q_c$$

LP-formulation for the general case

$$\max_p \min_q \sum_r p_r M[r,c] q_c = \max_p \min_c \sum_r p_r M[r,c]$$

$$= \max_p \max \{x : x \leq \sum_r p_r M[r,c] \ \forall c\}$$

$$= \boxed{\begin{array}{ll} \max & x \\ \text{s.t.} & x \leq (M^T p)_c \ \ \forall c \\ & \sum p_r = 1, \ p_r \geq 0 \end{array}}$$

(primal LP)

In a similar way, the min max is given by

$$\boxed{\begin{array}{ll} \min & y \\ \text{s.t.} & y \geq (Mq)_r \ \ \forall r \\ & \sum_c q_c = 1, \ \ q_c \geq 0 \end{array}}$$

(dual LP)