Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Fun Duality

a) Given the following LP, write out its dual and solve it by drawing out the feasible region.

$$\begin{aligned} \max\ & y - x \\ \text{s.t.}\ & x + 2y \leq 5 \\ & -x + y \leq 1 \\ & x, y \geq 0 \end{aligned}$$

Solution: Dual:

$$\begin{aligned} \min\ & 5y_1 + y_2 \\ \text{s.t.}\ & y_1 - y_2 \geq -1 \\ & 2y_1 + y_2 \geq 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

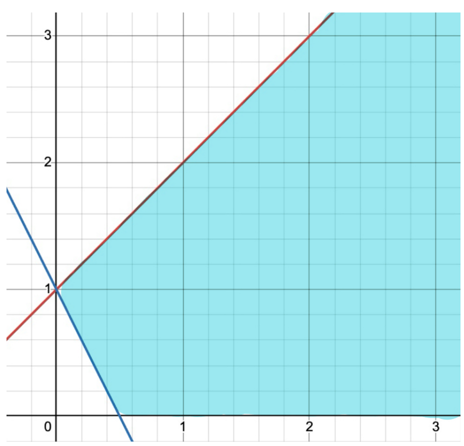


Figure 1: The light blue region is the feasible region.

To minimize the objective function, intuitively we should make y_1 as small as possible. Looking at the feasible region which is unbounded towards the right, we know the minimum must occur at the left most vertex which is (0, 1). Alternatively, you can try drawing out contour lines of the objective function and see that the objective function will be increasing as we move towards the right. Therefore, to minimize the objective value, we will choose the left most vertex which gives an objective value of 1.

b) Now use the optimum you found for dual LP in part a) to solve for the primal LP from part a). (Hint: what do we know about the relationship between primal LP's optimum and dual LP's optimum?)

Solution: From part a) we know the minimum objective value for dual LP is 1. By duality, we know that 1 will be an upperbound on solution to primal LP. We can then solve primal LP by trying out combinations of x and y to see whether there exists a combination such that y - x = 1 and at the same time all the constraints are satisfied. In this case we can see that (1,2) is a valid combination. And therefore, the optimum is achieved at vertex (1,2) with an objective value of 1.

c) For this subpart, assume the constraint $x + 2y \leq 5$ in primal LP is changed to $x + 2y = 5$. How would this change the dual LP?

Solution: The constraint $y_1, y_2 \geq 0$ will be changed to $y_2 \geq 0$. This is because we don't need to worry about flipping the inequality sign when we multiply a negative number to both sides of the equality. Thus, y_1 here no longer has to be non-negative.

d) This subpart is independent from the previous subparts. Given feasible (but not necessarily optimal!) solutions to the primal LP and dual LP respectively with the following objective values, describe the possible range of values that the true primal/dual optimal solution could have, or argue that it's not possible for the primal and dual LP solutions to take on the given values.

Assume primal is a maximization problem.

i) Primal: 10, Dual: 10

Solution: Only 10 is possible. Weak duality tells us that any feasible value of dual LP is an upper bound on the original primal LP. In this case, 10 is a tight upper bound.

ii) Primal: 10, Dual: 13

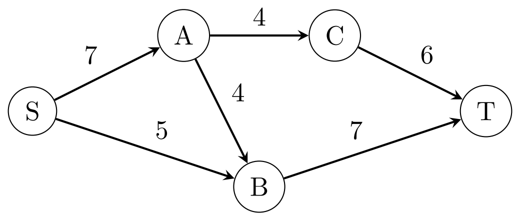
Solution: The optimal objective value will be somewhere in the interval $[10, 13]$.

iii) Primal: 10, Dual: 9

Solution: This scenario is not possible. It's impossible for primal LP to have a larger value than dual LP.

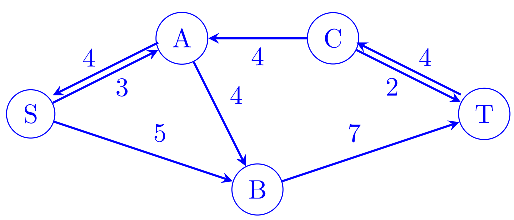
2 Residual in graphs

Consider the following graph with edge capacities as shown:



(a) Consider pushing 4 units of flow through $S \to A \to C \to T$. Draw the residual graph after this push.

Solution:

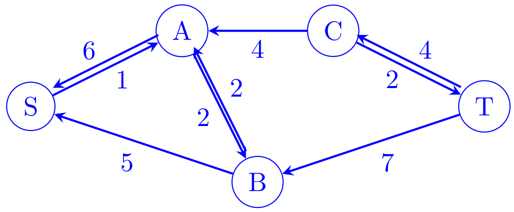


(b) Compute a maximum flow of the above graph. Find a minimum cut. Draw the residual graph of the maximum flow.

Solution: A maximum flow of value 11 results from pushing:

- 4 units of flow through $S \to A \to C \to T$;
- 5 units of flow through $S \to B \to T$; and
- 2 units of flow through $S \to A \to B \to T$.

(There are other maximum flows of the same value, can you find them?) The resulting residual graph (with respect to the maximum flow above) is:

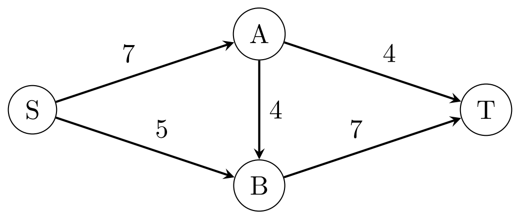


A minimum cut of value 11 is between $\{S, A, B\}$ and $\{C, T\}$ (with cross edges $A \to C$ and $B \to T$).

3 Max Flow, Min Cut, and Duality

In this exercise, we will demonstrate that LP duality can be used to show the max-flow min-cut theorem.

Consider this graph instance of max flow:



Let f_1 be the flow pushed on the path $\{S, A, T\}$, f_2 be the flow pushed on the path $\{S, A, B, T\}$, and f_3 be the flow pushed on the path $\{S, B, T\}$. The following is an LP for max flow in terms of the variables f_1, f_2, f_3:

$$
\begin{aligned}
\max \quad & f_1 + f_2 + f_3 & \\
& f_1 + f_2 \leq 7 & \text{(Constraint for } (S, A)\text{)} \\
& f_3 \leq 5 & \text{(Constraint for } (S, B)\text{)} \\
& f_1 \leq 4 & \text{(Constraint for } (A, T)\text{)} \\
& f_2 \leq 4 & \text{(Constraint for } (A, B)\text{)} \\
& f_2 + f_3 \leq 7 & \text{(Constraint for } (B, T)\text{)} \\
& f_1, f_2, f_3 \geq 0 &
\end{aligned}
$$

The objective is to maximize the flow being pushed, with the constraint that for every edge, we can't push more flow through that edge than its capacity allows.

(a) Find the dual of this linear program, where the variables in the dual are x_e for every edge e in the graph.

(b) Consider any cut in the graph. Show that setting $x_e = 1$ for every edge crossing this cut and $x_e = 0$ for every edge not crossing this cut gives a feasible solution to the dual program.

(c) Based on your answer to the previous part, what problem is being modelled by the dual program? By LP duality, what can you argue about this problem and the max flow problem?

Solution:

(a) The dual is:

$$\begin{aligned} \min \quad & 7x_{SA} + 5x_{SB} + 4x_{AT} + 4x_{AB} + 7x_{BT} \\ x_{SA} + x_{AT} &\geq 1 \qquad \text{(Constraint for } f_1\text{)} \\ x_{SA} + x_{AB} + x_{BT} &\geq 1 \qquad \text{(Constraint for } f_2\text{)} \\ x_{SB} + x_{BT} &\geq 1 \qquad \text{(Constraint for } f_3\text{)} \\ x_e &\geq 0 \quad \forall e \in E \end{aligned}$$

(b) Notice that each constraint contains all variables x_e for every edge e in the corresponding path. For any $s - t$ cut, every $s - t$ path contains an edge crossing this cut. So for any cut, the suggested solution will set at least one x_e to 1 on each path, giving that each constraint is satisfied.

(c) The dual LP is an LP for the min-cut problem. By the previous answer, we know the constraints describe solutions corresponding to cuts. The objective then just says to find the cut of the smallest size. By LP duality, the dual and primal optima are equal, i.e. the max flow and min cut values are equal.

4 Advertising

Pen & Perry's has a business competitor, Cold Pebble. Both of these dessert shops sell ice-pops, ice-creams, and ice-cakes. When both shops advertise the same thing, Cold Pebble gains an extra two customers. Otherwise, when Pen & Perry's advertises ice-pops, they gain one extra customer, and when they advertise ice-cakes, they gain three extra customers. In all other scenarios, neither shop gains or loses customers.

(a) Representing Pen & Perry's as the row and Cold Pebble as the column, what is the payoff matrix (for row's gains)?

Solution:

The payoff matrix is as follows, where p represents ice-pops, c represents ice-creams, and a represents ice-cakes.

	p	c	a
p	-2	1	1
c	0	-2	0
a	3	3	-2

(b) Suppose at the start of the day, Pen & Perry announces that there's a 20% chance they advertise ice-pops, 30% chance for ice-creams, and 50% chance for ice-cakes. What is Cold Pebble's optimal advertising strategy?

Solution: Cold Pebble loses $0.2(-2) + 0.3(0) + 0.5(3) = 1.1$ by advertising ice-pops, $0.2(1) + 0.3(-2) + 0.5(3) = 1.1$ by advertising ice-creams, and $0.2(1) + 0.3(0) + 0.5(-2) = -0.8$ by advertising ice-cakes. So Cold Pebble should always advertise ice-cakes.

(c) Write the corresponding LP for the optimal strategy for Pen & Perry's.

Solution:

$$\text{maximize } z$$

$$\text{subject to } \begin{cases} -2p + 3a \geq z \\ p - 2c + 3a \geq z \\ p - 2a \geq z \\ p + c + a = 1 \\ p, c, a, \geq 0 \end{cases}$$

(d) Write the corresponding LP for the optimal strategy for Cold Pebble.

Solution:

$$\text{minimize } z$$

$$\text{subject to } \begin{cases} -2p + c + a \leq z \\ -2c \leq z \\ 3p + 3c - 2a \leq z \\ p + c + a = 1 \\ p, c, a, \geq 0 \end{cases}$$

5 Zero-Sum Games Short Answer

(a) Suppose a zero-sum game has the following property: The payoff matrix M satisfies $M = -M^\top$. What is the expected payoff of the row player?

(b) True or False: If every entry in the payoff matrix is either 1 or -1 and the maximum number of 1s in any row is k, then for any row with less than k 1s, the row player's optimal strategy chooses this row with probability 0. Justify your answer.

(c) True or False: Let M_i denote the ith row of the payoff matrix. If $M_1 = \frac{M_2+M_3}{2}$, then there is an optimal strategy for the row player that chooses row 1 with probability 0.

Solution:

(a) To get the column player's payoff matrix, we negate the payoff matrix and take its transpose. So we get that the row and column players' payoff matrices are the same matrix. In turn, they must have the same expected payoff, but also the sum of their expected payoffs must be 0, so both players must have expected payoff 0.

(b) False: Consider the 2-by-3 payoff matrix:

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

The row player's optimal strategy is to choose the two rows with equal probability - note that the column player doesn't care about choosing column 1 vs column 3, so this game is no different than the zero-sum game for the 2-by-2 payoff matrix:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(c) True: Consider the optimal strategy for the row player. If the row player chooses rows 1, 2, 3 with probabilities p_1, p_2, p_3, they can instead choose row 1 with probability 0, row 2 with probability $p_2 + p_1/2$, and row 3 with probability $p_3 + p_1/2$. The expected payoff of this strategy is the same, so this strategy is also optimal.