Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Fun Duality

a) Given the following LP, write out its dual and solve it by drawing out the feasible region.

$$\begin{aligned} \max\ & y - x \\ \text{s.t. } & x + 2y \leq 5 \\ & -x + y \leq 1 \\ & x, y \geq 0 \end{aligned}$$

b) Now use the optimum you found for dual LP in part a) to solve for the primal LP from part a). (Hint: what do we know about the relationship between primal LP's optimum and dual LP's optimum?)

c) For this subpart, assume the constraint $x + 2y \leq 5$ in primal LP is changed to $x + 2y = 5$. How would this change the dual LP?

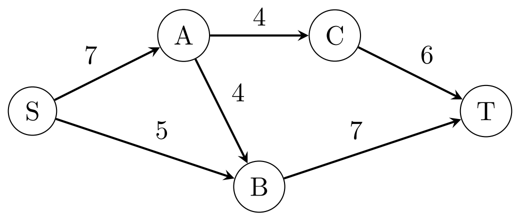
d) This subpart is independent from the previous subparts. Given feasible (but not necessarily optimal!) solutions to the primal LP and dual LP respectively with the following objective values, describe the possible range of values that the true primal/dual optimal solution could have, or argue that it's not possible for the primal and dual LP solutions to take on the given values.

Assume primal is a maximization problem.

i) Primal: 10, Dual: 10
ii) Primal: 10, Dual: 13
iii) Primal: 10, Dual: 9

2 Residual in graphs

Consider the following graph with edge capacities as shown:



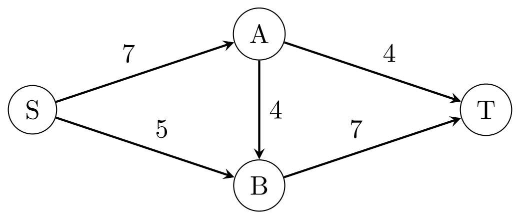
(a) Consider pushing 4 units of flow through $S \to A \to C \to T$. Draw the residual graph after this push.

(b) Compute a maximum flow of the above graph. Find a minimum cut. Draw the residual graph of the maximum flow.

3 Max Flow, Min Cut, and Duality

In this exercise, we will demonstrate that LP duality can be used to show the max-flow min-cut theorem.

Consider this graph instance of max flow:



Let f_1 be the flow pushed on the path $\{S, A, T\}$, f_2 be the flow pushed on the path $\{S, A, B, T\}$, and f_3 be the flow pushed on the path $\{S, B, T\}$. The following is an LP for max flow in terms of the variables f_1, f_2, f_3:

$$
\begin{aligned}
\max \quad & f_1 + f_2 + f_3 & \\
& f_1 + f_2 \leq 7 & \text{(Constraint for } (S, A)) \\
& f_3 \leq 5 & \text{(Constraint for } (S, B)) \\
& f_1 \leq 4 & \text{(Constraint for } (A, T)) \\
& f_2 \leq 4 & \text{(Constraint for } (A, B)) \\
& f_2 + f_3 \leq 7 & \text{(Constraint for } (B, T)) \\
& f_1, f_2, f_3 \geq 0 &
\end{aligned}
$$

The objective is to maximize the flow being pushed, with the constraint that for every edge, we can't push more flow through that edge than its capacity allows.

(a) Find the dual of this linear program, where the variables in the dual are x_e for every edge e in the graph.

(b) Consider any cut in the graph. Show that setting $x_e = 1$ for every edge crossing this cut and $x_e = 0$ for every edge not crossing this cut gives a feasible solution to the dual program.

(c) Based on your answer to the previous part, what problem is being modelled by the dual program? By LP duality, what can you argue about this problem and the max flow problem?

4 Advertising

Pen & Perry's has a business competitor, Cold Pebble. Both of these dessert shops sell ice-pops, ice-creams, and ice-cakes. When both shops advertise the same thing, Cold Pebble gains an extra two customers. Otherwise, when Pen & Perry's advertises ice-pops, they gain one extra customer, and when they advertise ice-cakes, they gain three extra customers. In all other scenarios, neither shop gains or loses customers.

(a) Representing Pen & Perry's as the row and Cold Pebble as the column, what is the payoff matrix (for row's gains)?

(b) Suppose at the start of the day, Pen & Perry announces that there's a 20% chance they advertise ice-pops, 30% chance for ice-creams, and 50% chance for ice-cakes. What is Cold Pebble's optimal advertising strategy?

(c) Write the corresponding LP for the optimal strategy for Pen & Perry's.

(d) Write the corresponding LP for the optimal strategy for Cold Pebble.

5 Zero-Sum Games Short Answer

(a) Suppose a zero-sum game has the following property: The payoff matrix M satisfies $M = -M^\top$. What is the expected payoff of the row player?

(b) True or False: If every entry in the payoff matrix is either 1 or -1 and the maximum number of 1s in any row is k, then for any row with less than k 1s, the row player's optimal strategy chooses this row with probability 0. Justify your answer.

(c) True or False: Let M_i denote the ith row of the payoff matrix. If $M_1 = \frac{M_2+M_3}{2}$, then there is an optimal strategy for the row player that chooses row 1 with probability 0.