CS 170 Homework 9

Due **3/22/2024, at 10:00 pm (grace period until 4/1/2024 11:59pm)**

1 Applications of Max-Flow Min-Cut

Review the statement of max-flow min-cut theorem and prove the following two statements.

(a) Let $G = (L \cup R, E)$ be a unweighted bipartite graph[1]. Then G has a L-perfect matching (a matching[2] with size $|L|$) if and only if, for every set $X \subseteq L$, X is connected to at least $|X|$ vertices in R. You must prove both directions.

Hint: Use the max-flow min-cut theorem on the cut that forms X and $L \setminus X$.

(b) Let G be an unweighted directed graph and $s, t \in V$ be two distinct vertices. Then the maximum number of edge-disjoint s-t paths equals the minimum number of edges whose removal disconnects t from s (*i.e.*, no directed path from s to t after the removal).

[1] A bipartite graph $G = (L \cup R, E)$ is defined as a graph that can be partitioned into two disjoint sets of vertices (i.e. L and R) such that no two vertices within the same set are adjacent.

[2] A matching is defined as a set of edges that share no common vertices.

2 The Matching Game

The matching game is played over the complete weighted bipartite graph $G(V, E)$ with positive edge weights w_e. The edge player plays an edge $e \in E$ while the vertex player plays a vertex $v \in V$ and if v is one of the endpoints of e (we will denote this by $v \in e$), the edge player pays w_e to the vertex player. The edge player would like to minimize the amount they have to pay the vertex player, while the vertex player wants to maximize their earnings.

(a) If the vertex player plays a uniformly random vertex what is the best response for the edge player?

(b) If the edge player plays a uniformly random edge from the minimum weight matching, what is the best response for the vertex player?

(c) Are these two strategies optimal for this game? If so, provide a brief justification. If not, write the 2 LP's corresponding to the edge player and vertex player, respectively, such that solving them yields each player's optimal strategy. You do not need to write these LP's in canonical form; however, it should be clear what all the constraints are.