LECTURE 22

4/16/24

SO FAR: 1) NP- Completeness Reductions

2) Approximation Algo

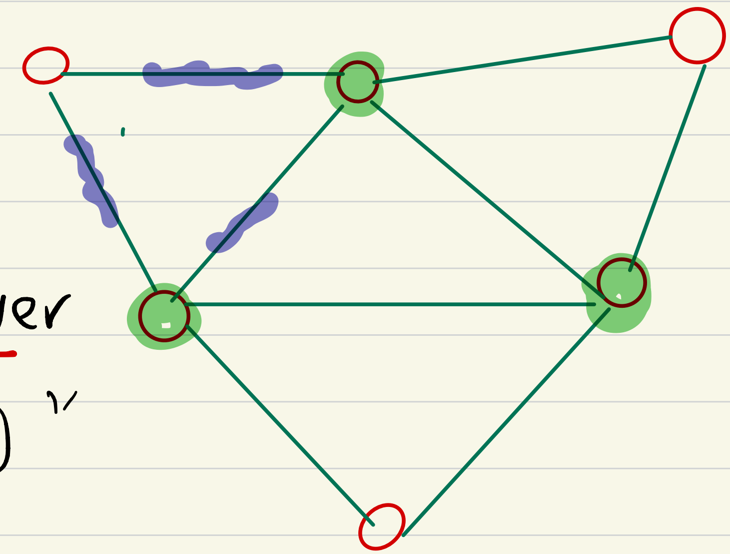
Ex.: Minimum Vertex Cover

TODAY: 1) Minimum Vertex Cover (Linear Program)

2) Metric TSP.

MINIMUM VERTEX COVER

INPUT: GRAPH $G=(V,E)$



DEF: A set $S \subseteq V$ is a vertex cover if $\forall$ edge (u,v) "S covers (u,v)"

$\Updownarrow$

at least one of $u,v \in S$.

SOL: Minimum size set S which is a vertex cover.

GOAL: A 2-approximation algo

$\iff \forall$ Graph G,

ALG-OUTPUT $\leq 2 \cdot$ Optimal Vertex Cover

LINEAR PROGRAM:

Variables:

For each vertex i

Intuition

$$x_i = \begin{cases} 0 & \text{if } i \notin \text{Vertex Cover} \\ 1 & \text{if } i \in \text{Vertex Cover} \end{cases}$$

MINIMUM VERTEX COVER

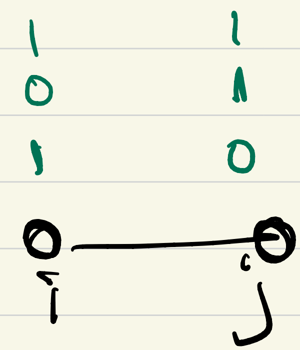
INPUT: Graph $G = (V, E)$

SOL: Smallest Vertex Cover S s.t. $\forall (u,v) \in E$, $u \in S$ or $v \in S$.

$$\text{Minimize} \sum_{i \in V} x_i \quad = \text{size of vertex cover}$$

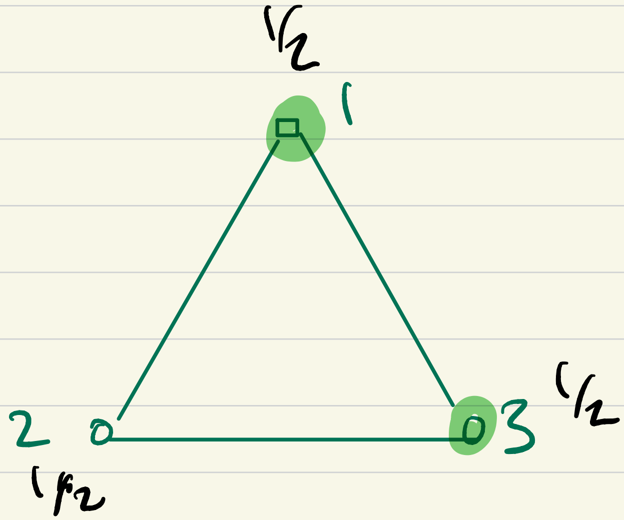
$$0 \leq x_i \leq 1 \quad \forall i \in V$$

$$x_i + x_j \geq 1 \quad \forall (i,j) \in E$$



(at least one of i or j is picked)

Output of LP = fractional solution!!



Optimal Vertex Cover = 2

$$\text{Min} \quad x_1 + x_2 + x_3$$

$$x_1 + x_2 \geq 1$$

$$x_2 + x_3 \geq 1$$

$$x_3 + x_1 \geq 1$$

$$0 \leq x_1, x_2, x_3 \leq 1$$

Set $x_1 = x_2 = x_3 = 1/2$

$$\text{OBJ} = 1/2 + 1/2 + 1/2 = 3/2 \ll 2$$

ROUNDING ALGORITHM:

Let x^* be the optimal LP-solution

$x_i^* \in [0,1] \ \forall i$

Pick:

$$S = \{ i \mid x_i^* \geq 1/2 \}$$

Claim: S covers all edges $(i,j) \in E$,

BECAUSE: $\forall (i,j) \in E \quad x_i^* + x_j^* \geq 1$

$\Rightarrow$ At least one of $x_i^*, x_j^* \geq 1/2$

$\Rightarrow$ At least one of i or $j \in S$.

Claim: $|S| \leq 2 \cdot$ OPTIMAL VERTEX COVER

BECAUSE

$$|S| \leq 2 \cdot \left(\text{Cost of LP} = \sum_i x_i\right)$$

(linear program)

Consider any vertex $i \in S$

vertex i costs 1 on LHS

vertex i contributes $x_i^* \geq 1/2$ to LP OPT.

$$\text{Cost of LP} = \sum_{i \in V} x_i^*$$

$$\geq \sum_{i \in S} x_i^*$$

$$\geq \sum_{i \in S} (1/2)$$

[BECAUSE $x_i^* \geq 1/2$ $\forall\, i \in S$]

$$\text{Cost of LP} \geq |S|/2$$

$$\text{Cost of LP} \leq \text{Cost of OPTIMAL VERTEX COVER}$$

PROOF: BECAUSE Every Vertex Cover

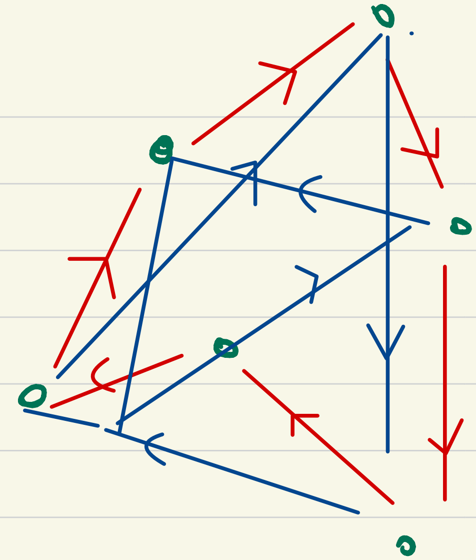
$\Rightarrow$ is also a solution to linear program

But linear program gets to choose fractional values

ALG OUTPUT $\leq 2 \cdot$ (Cost of LP)

$\leq 2 \cdot$ (Cost of Optimal Sol)

(METRIC) TRAVELLING SALESMAN PROBLEM

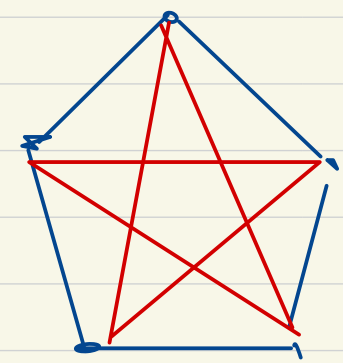


INPUT: n cities,

Distances $d(i,j)$ between i and j

SOL: Shortest tour that visits all vertices and returns back.

(once)

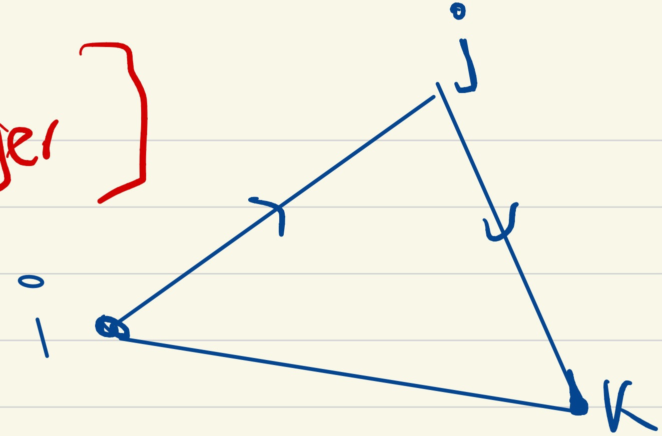


ASSUMPTION: 1) Every $i \to j$ edge exists

(METRIC / TRIANGLE INEQUALITY)

$$d(i,j) + d(j,k) \geq d(i,k) \quad \forall i,j,k$$

[Direct routes are no longer than other routes]



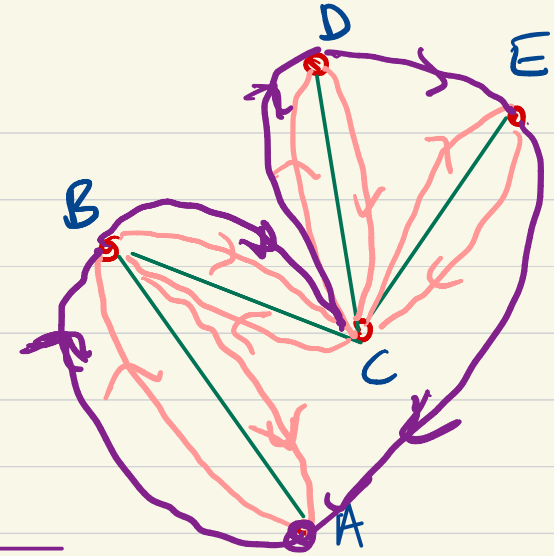
Theorem:

TSP (without Δle inequality) cannot be approximated to any factor unless $NP = P$.

Is there a better than 2-approximation for Vertex Cover? (Assume $NP \neq P$)

APPROX ALG:

STEP I: Find a MST T



$$\text{COST}(T) \leq \text{Cost}(\text{OPTIMAL TSP TOUR})$$

STEP 2: Find a Depth First Traversal of T

$$A \to B \to C \to D \to C \to E \to C \to B \to A$$

$$\text{COST}(\text{DFS TRAVERSAL}) = 2 \cdot \text{COST}(\text{Tree})$$

STEP 3: Skip ALL repeated vertices

$$A \to B \to C \to D \to C \to E \to C \to B \to A$$

$$\text{COST}(\text{OUTPUT TOUR}) \leq \text{COST}(\text{DFS TRAVERSAL})$$