Today.

...Complex numbers, polynomials today. FFT.

Multiplying polynomials.

$$(1+2x+3x^2)(4+3x+2x^2)$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Coefficient of x^4 in result?

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Coefficient of x^4 in result?

(A) 6

(B) 5

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Coefficient of x^4 in result?

(A) 6

(B) 5

(A) 6

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Coefficient of x^4 in result?

(A) 6

(B) 5

(A) 6 of course!

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Coefficient of x^4 in result?

(A) 6

(B) 5

(A) 6 of course!

Coeefficient of x^2 in result?

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Coefficient of x^4 in result?

(A) 6

(B) 5

(A) 6 of course!

Coeefficient of x^2 in result?

Uh oh...

Multiplying polynomials.

$$(1+2x+3x^2)(4+3x+2x^2)$$

Multiplying polynomials.

$$(1+2x+3x^2)(4+3x+2x^2)$$

$$x^0$$

Multiplying polynomials.

$$(1 + 2x + 3x^2)(4 + 3x + 2x^2)$$

$$x^0 \quad ((1)(4))$$

Multiplying polynomials.

$$(1+2x+3x^2)(4+3x+2x^2)$$

$$x^0 \quad ((1)(4)) \qquad = 4$$

Multiplying polynomials.

$$(1+2x+3x^2)(4+3x+2x^2)$$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & & \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3) & \end{array}$$

Multiplying polynomials.

$$(1 + 2x + 3x^2)(4 + 3x + 2x^2)$$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3) + (2)(4)) & \end{array}$$

Multiplying polynomials.

$$(1+2x+3x^2)(4+3x+2x^2)$$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \end{array}$$

Multiplying polynomials.

$$(1+2x+3x^2)(4+3x+2x^2)$$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & & \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2) & \end{array}$$

Multiplying polynomials.

$$(1+2x+3x^2)(4+3x+2x^2)$$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3) & \end{array}$$

Multiplying polynomials.

$(1 + 2x + 3x^2)(4 + 3x + 2x^2)$

$$
\begin{array}{lll}
x^0 & ((1)(4)) & = 4 \\
x^1 & ((1)(3) + (2)(4)) & = 11 \\
x^2 & ((1)(2) + (2)(3) + (3)(4))) &
\end{array}
$$

Multiplying polynomials.

$$(1 + 2x + 3x^2)(4 + 3x + 2x^2)$$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3) + (2)(4)) & = 11 \\ x^2 & ((1)(2) + (2)(3) + (3)(4))) & = 20 \end{array}$$

Multiplying polynomials.

$$(1+2x+3x^2)(4+3x+2x^2)$$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\ x^3 & & \end{array}$$

Multiplying polynomials.

$(1 + 2x + 3x^2)(4 + 3x + 2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3) + (2)(4)) & = 11 \\ x^2 & ((1)(2) + (2)(3) + (3)(4))) & = 20 \\ x^3 & ((2)(2) & \end{array}$$

Multiplying polynomials.

$$(1+2x+3x^2)(4+3x+2x^2)$$

$$\begin{array}{lll} x^0 & ((1)(4)) & =4 \\ x^1 & ((1)(3)+(2)(4)) & =11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & =20 \\ x^3 & ((2)(2)+(3)(3)) & \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\ x^3 & ((2)(2)+(3)(3)) & = 13 \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$
\begin{array}{lll}
x^0 & ((1)(4)) & = 4 \\
x^1 & ((1)(3)+(2)(4)) & = 11 \\
x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\
x^3 & ((2)(2)+(3)(3)) & = 13 \\
x^4 & &
\end{array}
$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$
\begin{array}{lll}
x^0 & ((1)(4)) & =4 \\
x^1 & ((1)(3)+(2)(4)) & =11 \\
x^2 & ((1)(2)+(2)(3)+(3)(4))) & =20 \\
x^3 & ((2)(2)+(3)(3)) & =13 \\
x^4 & ((3)(2)) &
\end{array}
$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$
\begin{array}{lll}
x^0 & ((1)(4)) & = 4 \\
x^1 & ((1)(3)+(2)(4)) & = 11 \\
x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\
x^3 & ((2)(2)+(3)(3)) & = 13 \\
x^4 & ((3)(2)) & = 6
\end{array}
$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\ x^3 & ((2)(2)+(3)(3)) & = 13 \\ x^4 & ((3)(2)) & = 6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & =4 \\ x^1 & ((1)(3)+(2)(4)) & =11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & =20 \\ x^3 & ((2)(2)+(3)(3)) & =13 \\ x^4 & ((3)(2)) & =6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Given:

$a_0+a_1x+\cdots a_dx^d$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\ x^3 & ((2)(2)+(3)(3)) & = 13 \\ x^4 & ((3)(2)) & = 6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Given:

$a_0+a_1x+\cdots a_dx^d$ In example: $a_0=1, a_1=2, a_2=3$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & =4 \\ x^1 & ((1)(3)+(2)(4)) & =11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & =20 \\ x^3 & ((2)(2)+(3)(3)) & =13 \\ x^4 & ((3)(2)) & =6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Given:

$a_0+a_1x+\cdots a_dx^d$ In example: $a_0=1, a_1=2, a_2=3$

$b_0+b_1x+\cdots b_dx^d$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\ x^3 & ((2)(2)+(3)(3)) & = 13 \\ x^4 & ((3)(2)) & = 6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Given:

$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$

$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\ x^3 & ((2)(2)+(3)(3)) & = 13 \\ x^4 & ((3)(2)) & = 6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Given:

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$b_0+b_1x+\cdots b_dx^d$ In example: $b_0=4, b_1=3, b_2=2$

Product: $c_0+c_1x+\cdots c_{2d}x^{2d}$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & =4 \\ x^1 & ((1)(3)+(2)(4)) & =11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & =20 \\ x^3 & ((2)(2)+(3)(3)) & =13 \\ x^4 & ((3)(2)) & =6 \end{array}$$

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Product: $c_0+c_1x+\cdots c_{2d}x^{2d}$

$$c_k = \sum_{0\le i\le k} a_i * b_{k-i}.$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & =4 \\ x^1 & ((1)(3)+(2)(4)) & =11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & =20 \\ x^3 & ((2)(2)+(3)(3)) & =13 \\ x^4 & ((3)(2)) & =6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Given:

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Product: $c_0+c_1x+\cdots c_{2d}x^{2d}$

$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2b_0 + a_1b_1 + a_0b_2$.

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:

$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
Product: $c_0 + c_1 x + \cdots c_{2d} x^{2d}$

$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

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Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:

$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
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$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$.
Runtime?

(A) $O(d)$

(B) $O(d \log d)$

(C) $O(n^2)$

(D) $O(d^2)$

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:
$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
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Runtime?

(A) $O(d)$

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(C) $O(n^2)$

(D) $O(d^2)$

Time: $O(k)$ multiplications for each k up to $k = 2d$.

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:
$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
Product: $c_0 + c_1 x + \cdots c_{2d} x^{2d}$

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Runtime?

(A) $O(d)$

(B) $O(d \log d)$

(C) $O(n^2)$

(D) $O(d^2)$

Time: $O(k)$ multiplications for each k up to $k = 2d$.
$\implies O(d^2)$.

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:
$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
Product: $c_0 + c_1 x + \cdots c_{2d} x^{2d}$

$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$.
Runtime?

(A) $O(d)$

(B) $O(d \log d)$

(C) $O(n^2)$

(D) $O(d^2)$

Time: $O(k)$ multiplications for each k up to $k = 2d$.
$\implies O(d^2)$.
or (D)

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:
$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
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Product: $c_0 + c_1 x + \cdots c_{2d} x^{2d}$

$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$.
Runtime?

(A) $O(d)$

(B) $O(d \log d)$

(C) $O(n^2)$

(D) $O(d^2)$

Time: $O(k)$ multiplications for each k up to $k = 2d$.
$\implies O(d^2)$.
or (D) ..will use n as parameter shortly.

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:
$a_0 + a_1x + \cdots a_dx^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1x + \cdots b_dx^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
Product: $c_0 + c_1x + \cdots c_{2d}x^{2d}$

$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2b_0 + a_1b_1 + a_0b_2$.
Runtime?

(A) $O(d)$

(B) $O(d \log d)$

(C) $O(n^2)$

(D) $O(d^2)$

Time: $O(k)$ multiplications for each k up to $k = 2d$.
$\implies O(d^2)$.
or (D) ..will use n as parameter shortly. so (C) also.

Hmmm...

$O(d^2)$ time!

Hmmm...

$O(d^2)$ time!
Quadratic Time!

Hmmm...

$O(d^2)$ time!
Quadratic Time!
Can we do better?

Hmmm...

$O(d^2)$ time!
Quadratic Time!
Can we do better?
Yes?

Hmmm...

$O(d^2)$ time!
Quadratic Time!
Can we do better?
Yes? No?

Hmmm...

$O(d^2)$ time!
Quadratic Time!
Can we do better?
Yes? No?
How?

Hmmm...

$O(d^2)$ time!
Quadratic Time!
Can we do better?
Yes? No?
How?
Use different representation.

Another representation.

Represent a line?

Another representation.

Represent a line?
Slope and intercept!

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line?

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)?

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

$d + 1$

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

$d + 1$

How to find points on function?

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

$d + 1$

How to find points on function?
plug in x-values...

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

$d + 1$

How to find points on function?
plug in x-values...and evaluate.

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

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$d + 1$

How to find points on function?
plug in x-values...and evaluate.

How to find “line” from points?

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

$d + 1$

How to find points on function?
plug in x-values...and evaluate.

How to find "line" from points?
Solve two variable system of equations!

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

$d + 1$

How to find points on function?
plug in x-values...and evaluate.

How to find “line” from points?
Solve two variable system of equations!

How to find polynomial from points?

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

$d + 1$

How to find points on function?
plug in x-values...and evaluate.

How to find "line" from points?
Solve two variable system of equations!

How to find polynomial from points?
Solve $d + 1$ variable system of equations!

Point-value representation.

$A(x_0), \cdots, A(x_{2d})$
$B(x_0), \cdots, B(x_{2d})$

Point-value representation.

$A(x_0), \cdots, A(x_{2d})$
$B(x_0), \cdots, B(x_{2d})$

Product: $C(x_0), \cdots, C(x_{2d})$

Point-value representation.

$A(x_0), \cdots, A(x_{2d})$
$B(x_0), \cdots, B(x_{2d})$

Product: $C(x_0), \cdots, C(x_{2d})$

$$C(x_i) = A(x_i)B(x_i)$$

Point-value representation.

$A(x_0), \cdots, A(x_{2d})$
$B(x_0), \cdots, B(x_{2d})$

Product: $C(x_0), \cdots, C(x_{2d})$

$$C(x_i) = A(x_i)B(x_i)$$

$O(d)$ multiplications!

Point-value representation.

$A(x_0), \cdots, A(x_{2d})$
$B(x_0), \cdots, B(x_{2d})$

Product: $C(x_0), \cdots, C(x_{2d})$

$$C(x_i) = A(x_i)B(x_i)$$

$O(d)$ multiplications!

Given: $a_0, \ldots, a_d$ and $b_0, \ldots, b_d$.

Point-value representation.

$A(x_0), \cdots, A(x_{2d})$
$B(x_0), \cdots, B(x_{2d})$

Product: $C(x_0), \cdots, C(x_{2d})$

$$C(x_i) = A(x_i)B(x_i)$$

$O(d)$ multiplications!

Given: $a_0, \ldots, a_d$ and $b_0, \ldots, b_d$.
Evaluate: $A(x), B(x)$ on $2d + 1$ points: $x_0, \cdots, x_{2d}$.

Point-value representation.

$A(x_0), \cdots, A(x_{2d})$
$B(x_0), \cdots, B(x_{2d})$

Product: $C(x_0), \cdots, C(x_{2d})$

$$C(x_i) = A(x_i)B(x_i)$$

$O(d)$ multiplications!

Given: $a_0, \ldots, a_d$ and $b_0, \ldots, b_d$.
Evaluate: $A(x), B(x)$ on $2d+1$ points: $x_0, \cdots, x_{2d}$.
Recall(from CS70): unique representation of polynomial.

Point-value representation.

$A(x_0), \cdots, A(x_{2d})$
$B(x_0), \cdots, B(x_{2d})$

Product: $C(x_0), \cdots, C(x_{2d})$

$$C(x_i) = A(x_i)B(x_i)$$

$O(d)$ multiplications!

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Interpolate: find $c_0 + c_1 x + c_2 x^2 + \cdots c_{2d} x^{2d}$.

Interpolation

Points: $(x_0, y_0), \ldots (x_d, y_d)$.

Interpolation

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Lagrange:

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Points: $(x_0, y_0), \ldots (x_d, y_d)$.

Lagrange:

$$\Delta_i(x) = \Pi_{i \neq j} \frac{x - x_j}{x_i - x_j}$$

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Points: $(x_0, y_0), \ldots (x_d, y_d)$.

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$\Delta_i(x) = \Pi_{i \neq j} \frac{x - x_j}{x_i - x_j}$

$P(x) = \sum_i y_i \Delta_i(x)$.

Interpolation

Points: $(x_0, y_0), \ldots (x_d, y_d)$.

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Correctness: $\Delta_i(x_j) = 0$ for $x_i \neq x_j$ and $\Delta_i(x_i) = 1$.

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Linear system:

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Points: $(x_0, y_0), \ldots (x_d, y_d)$.

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Linear system:

$c_0 + c_1 x_0 + c_2 x_0^2 \cdots c_d x_0^d = y_0$.

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Interpolation

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Has solution?

Interpolation

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Has solution? Lagrange.

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Unique?

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Unique?
At most d roots in any degree d polynomial.

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Not unique $\implies$ $P(x)$ and $Q(x)$ where $P(x_i) = Q(x_i)$.

Interpolation

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$P(x) - Q(x)$ has $d + 1$ roots.

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At most d roots in any degree d polynomial.
Not unique $\implies P(x)$ and $Q(x)$ where $P(x_i) = Q(x_i)$.
$P(x) - Q(x)$ has $d+1$ roots. Contradicts not unique.

What is it good for?

What is the point-value representation good for (from CS70)?

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Error tolerance.

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Can lose *any* k points and reconstruct.

The original "message/file/polynomial" is recoverable.

Polynomial Evaluation.

Evaluate $A(x) = a_0 + a_1 x + \cdots a_{n-1} x^{n-1}$ on n points: $x_0, \cdots, x_{n-1}$.

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Example: $4 + 3x + 5x^2 + 4x^3$ on 2.

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In general: $a_0 + x(a_1 + x(a_2 + x(...)))$.

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n multiplications/additions to evaluate one point.

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Evaluate on n points . We get $O(n^2)$ time.

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In general: $a_0 + x(a_1 + x(a_2 + x(...)))$.

n multiplications/additions to evaluate one point.

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Could have just multiplied polynomials!

Evaluation of polynomials: Recursive.

$$A(x) = A_e(x^2) + x(A_o(x^2))$$

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$$A(x) = 4 + 12x + 20x^2 + 13x^3 + 6x^4 + 7x^5$$

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$$\begin{aligned} A(x) &= 4 + 12x + 20x^2 + 13x^3 + 6x^4 + 7x^5 \\ &= (4 + 20x^2 + 6x^4) + (12x + 13x^3 + 7x^5) \end{aligned}$$

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Example:

$$\begin{aligned} A(x) &= 4 + 12x + 20x^2 + 13x^3 + 6x^4 + 7x^5 \\ &= (4 + 20x^2 + 6x^4) + (12x + 13x^3 + 7x^5) \\ &= (4 + 20x^2 + 6x^4) + x(12 + 13x^2 + 7x^4) \end{aligned}$$

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Evaluation of polynomials: Recursive.

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Plug in x^2 into A_e and A_o use results to find $A(x)$.

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Evaluate recursively:
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No better than polynomial multiplication! Bummer.

Recursive on more than one point.

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$\frac{n}{2}$ points: squares should only be $n/4$ distinct numbers

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How can squares be negative?

Explore recursion.

Recursive Condition:
n points: $\frac{n}{2}$ pairs of distinct numbers with common squares.
E.g., $\pm x_0$ both have x_0^2 as square,
$\pm x_1$ both have x_1^2 as square.
$\cdots$..

Next step:
$\frac{n}{2}$ points: squares should only be $n/4$ distinct numbers

But all our $\frac{n}{2}$ points are squares ...and positive!

Need the $\frac{n}{2}$ points to come in Positive/Negatives pairs!
How can squares be negative?

Complex numbers!

Pairs with common squares.

Want n numbers:

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}|$

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
and

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
and
$|\{x_0^4, \ldots, x_{n-1}^4\}|$

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
and
$|\{x_0^4, \ldots, x_{n-1}^4\}| = \frac{n}{4}$,

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
and
$|\{x_0^4, \ldots, x_{n-1}^4\}| = \frac{n}{4}$,
...and ...

Pairs with common squares.

Want n numbers:

$x_0, \ldots, x_{n-1}$ where

$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,

and

$|\{x_0^4, \ldots, x_{n-1}^4\}| = \frac{n}{4}$,

...and ...

$\{x_0^{\log n}, \ldots, x_{n-1}^{\log n}\}|$

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2},$$
and
$$|\{x_0^4, \ldots, x_{n-1}^4\}| = \frac{n}{4},$$
...and ...
$$\{x_0^{\log n}, \ldots, x_{n-1}^{\log n}\}| = 1.$$

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
and
$|\{x_0^4, \ldots, x_{n-1}^4\}| = \frac{n}{4}$,
...and ...
$\{x_0^{\log n}, \ldots, x_{n-1}^{\log n}\}| = 1$.

Each recursive level evaluates:

Pairs with common squares.

Want n numbers:

$x_0, \ldots, x_{n-1}$ where

$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,

and

$|\{x_0^4, \ldots, x_{n-1}^4\}| = \frac{n}{4}$,

...and ...

$\{x_0^{\log n}, \ldots, x_{n-1}^{\log n}\}| = 1$.

Each recursive level evaluates:

polynomials of half the degree on half as many points.

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
and
$|\{x_0^4, \ldots, x_{n-1}^4\}| = \frac{n}{4}$,
...and ...
$\{x_0^{\log n}, \ldots, x_{n-1}^{\log n}\}| = 1$.

Each recursive level evaluates:
polynomials of half the degree on half as many points.
n represents both degree and number of points.

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
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$\{x_0^{\log n}, \ldots, x_{n-1}^{\log n}\}| = 1$.

Each recursive level evaluates:
polynomials of half the degree on half as many points.
n represents both degree and number of points.

In reverse: start with a number 1

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
and
$|\{x_0^4, \ldots, x_{n-1}^4\}| = \frac{n}{4}$,
...and ...
$\{x_0^{\log n}, \ldots, x_{n-1}^{\log n}\}| = 1$.

Each recursive level evaluates:
polynomials of half the degree on half as many points.
n represents both degree and number of points.

In reverse: start with a number 1
Take square roots: 1, -1.

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
and
$|\{x_0^4, \ldots, x_{n-1}^4\}| = \frac{n}{4}$,
...and ...
$\{x_0^{\log n}, \ldots, x_{n-1}^{\log n}\}| = 1$.

Each recursive level evaluates:
polynomials of half the degree on half as many points.
n represents both degree and number of points.

In reverse: start with a number 1
Take square roots: 1, -1.
Take square roots: 1,-1, i, $-i$.

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
and
$|\{x_0^4, \ldots, x_{n-1}^4\}| = \frac{n}{4}$,
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Each recursive level evaluates:
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In reverse: start with a number 1
Take square roots: 1, -1.
Take square roots: 1,-1, i, $-i$.
Uh oh.

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
and
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Each recursive level evaluates:
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n represents both degree and number of points.

In reverse: start with a number 1
Take square roots: 1, -1.
Take square roots: 1,-1, i, $-i$.
Uh oh.

Actually: ± 1, $\pm i$, $\pm\frac{1}{\sqrt{2}}(1+i)$, $\pm\frac{1}{\sqrt{2}}(-1+i)$,

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
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Actually: ± 1, $\pm i$, $\pm\frac{1}{\sqrt{2}}(1+i)$, $\pm\frac{1}{\sqrt{2}}(-1+i)$,

Complex numbers!

Pairs with common squares.

Want n numbers:
$x_0, \ldots, x_{n-1}$ where
$|\{x_0^2, \ldots, x_{n-1}^2\}| = \frac{n}{2}$,
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$\{x_0^{\log n}, \ldots, x_{n-1}^{\log n}\}| = 1$.

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In reverse: start with a number 1
Take square roots: 1, -1.
Take square roots: 1,-1, i, $-i$.
Uh oh.

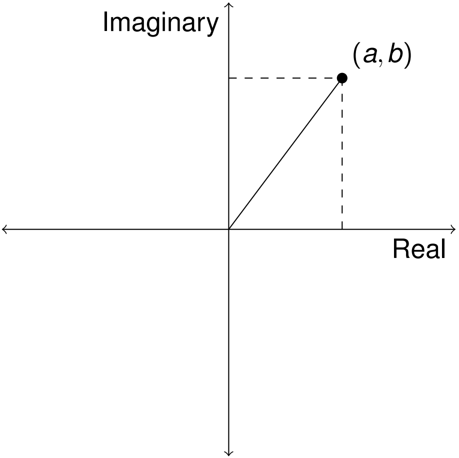
Actually: ± 1, $\pm i$, $\pm \frac{1}{\sqrt{2}}(1+i)$, $\pm \frac{1}{\sqrt{2}}(-1+i)$,

Complex numbers!

Uh oh? Can we get a pattern?

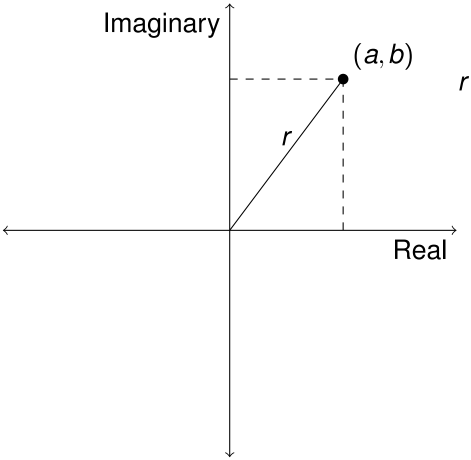
Complex plane

$z = a + bi$



Complex plane

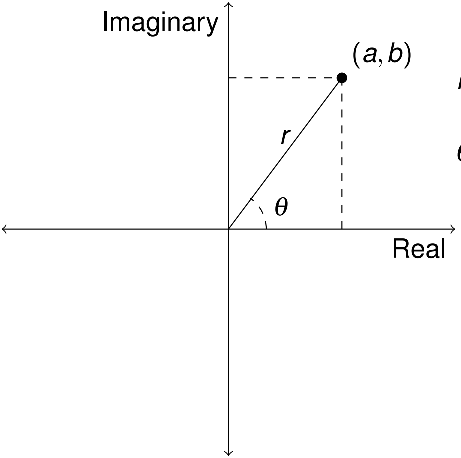
$$z = a + bi$$



$$r = \sqrt{a^2 + b^2}$$

Complex plane

$$z = a + bi$$

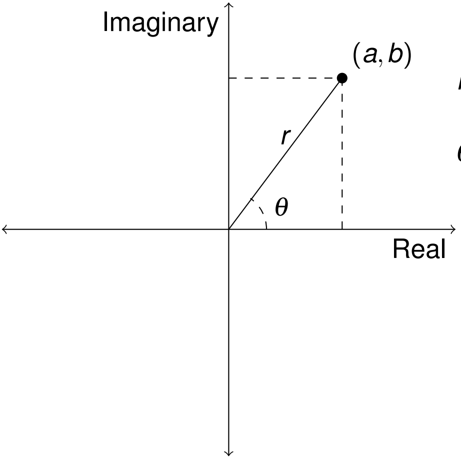


$$r = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}(\frac{b}{a})$$

Complex plane

$z = a + bi$

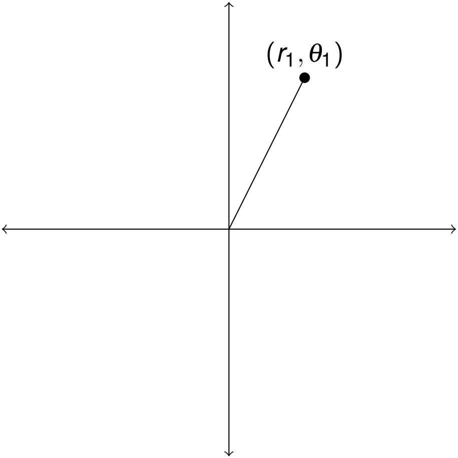


$r = \sqrt{a^2 + b^2}$

$\theta = \tan^{-1}(\frac{b}{a})$

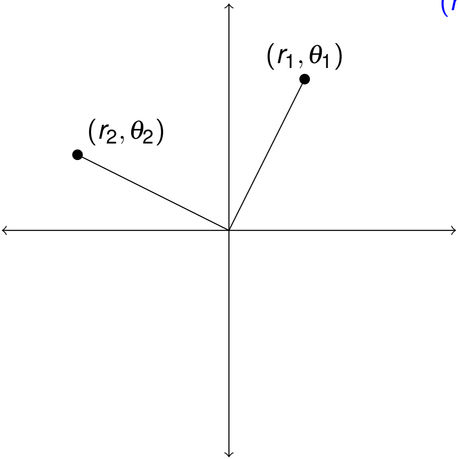
Polar coordinate: $r(\cos\theta + i\sin\theta) = re^{i\theta}$ or (r, θ)

Multiplying Complex Numbers

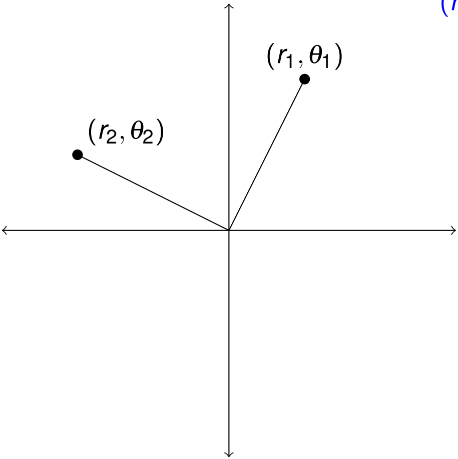


Multiplying Complex Numbers

$(r_1, \theta_1) \times (r_2, \theta_2)$



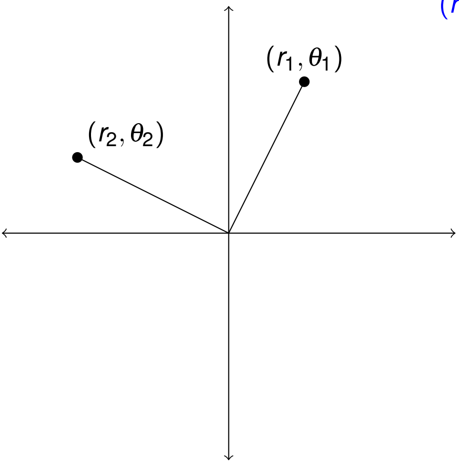
Multiplying Complex Numbers



$(r_1, \theta_1) \times (r_2, \theta_2)$

$= r_1 e^{i\theta_1} \times r_2 e^{i\theta_2}$

Multiplying Complex Numbers

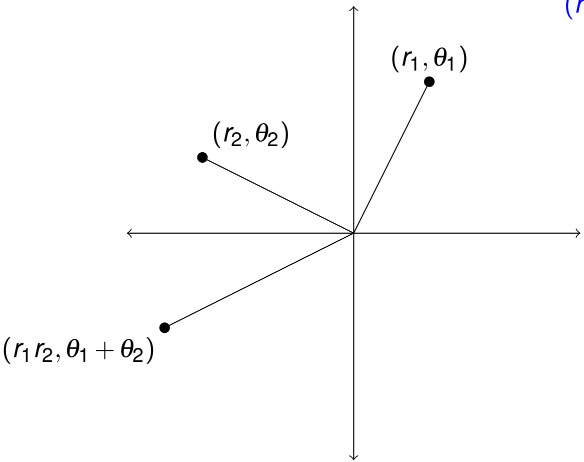


$(r_1, \theta_1) \times (r_2, \theta_2)$

$= r_1 e^{i\theta_1} \times r_2 e^{i\theta_2}$

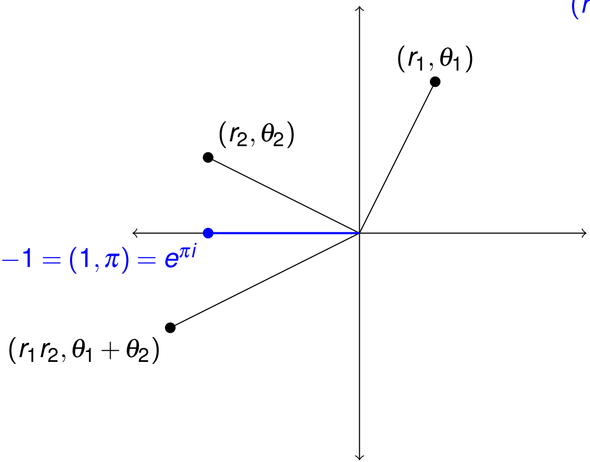
$= (r_1 r_2) e^{i(\theta_1 + \theta_2)}$

Multiplying Complex Numbers



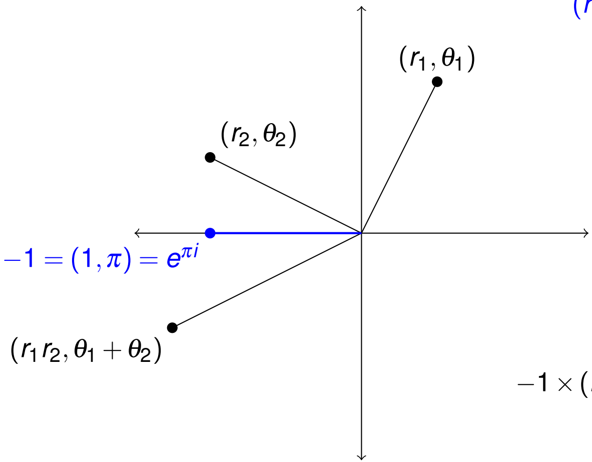
$$
\begin{aligned}
&(r_1, \theta_1) \times (r_2, \theta_2) \\
&\quad = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} \\
&\quad = (r_1 r_2) e^{i(\theta_1 + \theta_2)} \\
&\quad = (r_1 r_2, \theta_1 + \theta_2)
\end{aligned}
$$

Multiplying Complex Numbers



$$
\begin{aligned}
(r_1, \theta_1) \times (r_2, \theta_2) \\
&= r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} \\
&= (r_1 r_2) e^{i(\theta_1 + \theta_2)} \\
&= (r_1 r_2, \theta_1 + \theta_2)
\end{aligned}
$$

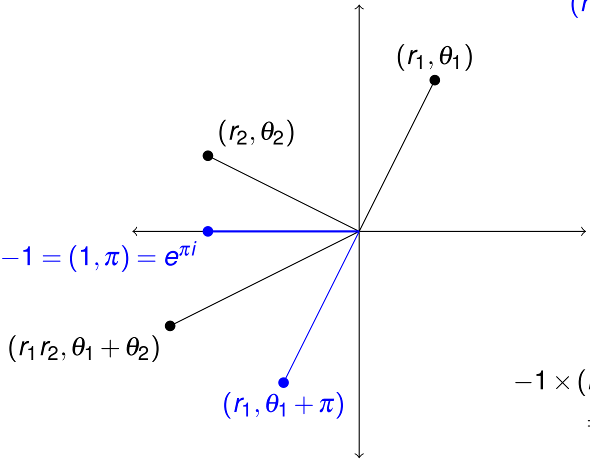
Multiplying Complex Numbers



$$(r_1, \theta_1) \times (r_2, \theta_2)$$
$$= r_1 e^{i\theta_1} \times r_2 e^{i\theta_2}$$
$$= (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$
$$= (r_1 r_2, \theta_1 + \theta_2)$$

$$-1 \times (r_1, \theta_1) = (r_1, \theta_1 + \pi)$$

Multiplying Complex Numbers



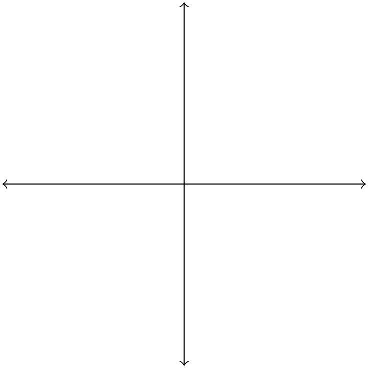
$$
\begin{aligned}
&(r_1, \theta_1) \times (r_2, \theta_2) \\
&\quad = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} \\
&\quad = (r_1 r_2) e^{i(\theta_1 + \theta_2)} \\
&\quad = (r_1 r_2, \theta_1 + \theta_2)
\end{aligned}
$$

$$
\begin{aligned}
-1 \times (r_1, \theta_1) &= (r_1, \theta_1 + \pi) \\
&= r_1 e^{r_1(\theta_1 + \pi)i}
\end{aligned}
$$

The *n*th complex roots of unity.

$$(e^{\frac{2i\pi}{n}+\pi})^2 = (e^{\frac{2i\pi}{n}})^2 e^{2\pi} = (e^{\frac{2i\pi}{n}})^2$$

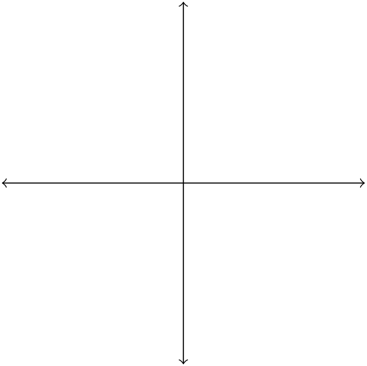
Solutions to $z^n = 1$



The nth complex roots of unity.

Solutions to $z^n = 1$

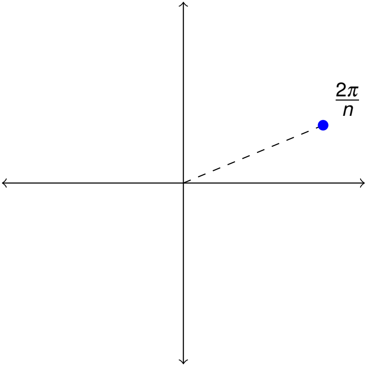
$$(1, \frac{2\pi}{n})^n = (1, \frac{2\pi}{n} \times n) = (1, 2\pi) = 1!$$



The nth complex roots of unity.

Solutions to $z^n = 1$

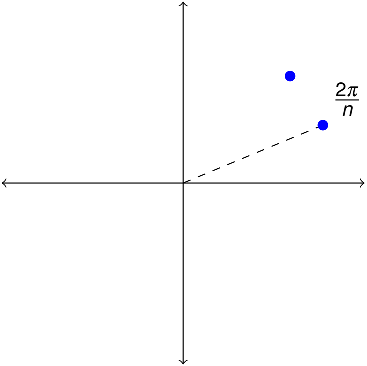
$$(1, \frac{2\pi}{n})^n = (1, \frac{2\pi}{n} \times n) = (1, 2\pi) = 1!$$



The nth complex roots of unity.

Solutions to $z^n = 1$

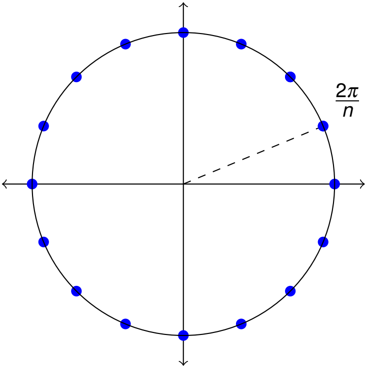
$$(1, \frac{4\pi}{n})^n = (1, \frac{4\pi}{n} \times n) = (1, 4\pi) = 1!$$



The nth complex roots of unity.

Solutions to $z^n = 1$

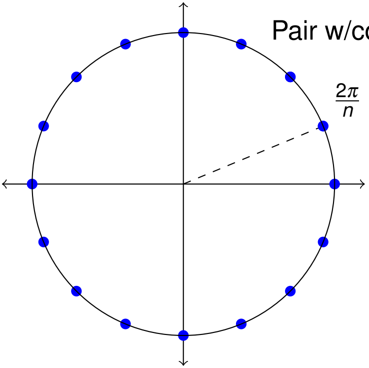
$$(1, \frac{2k\pi}{n})^n = (1, \frac{2k\pi}{n} \times n) = (1, 2k\pi) = 1!$$



The nth complex roots of unity.

Solutions to $z^n = 1$

$$(1, \theta + \pi)^2 = (1, 2\theta + 2\pi) = \boxed{(1, 2\theta)} = (1, \theta)^2.$$

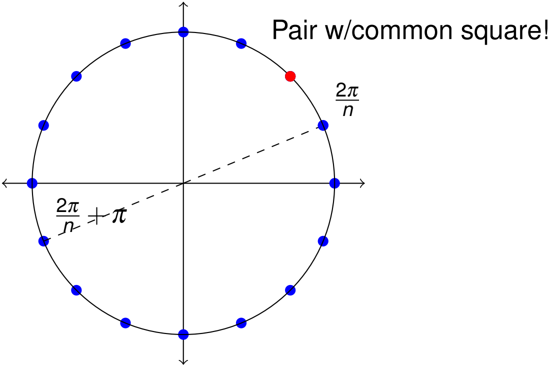


Pair w/common square!

The nth complex roots of unity.

Solutions to $z^n = 1$

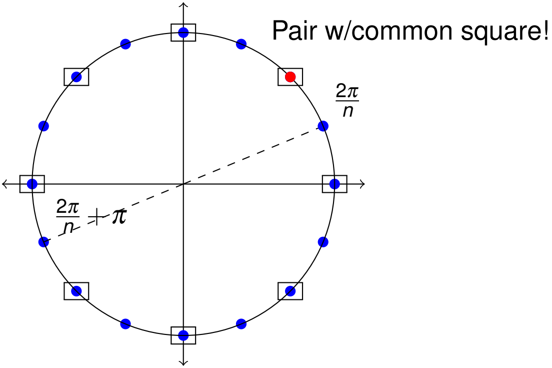
$$(1, \theta + \pi)^2 = (1, 2\theta + 2\pi) = \boxed{(1, 2\theta)} = (1, \theta)^2.$$



The nth complex roots of unity.

Solutions to $z^n = 1$

$$(1, \theta + \pi)^2 = (1, 2\theta + 2\pi) = \boxed{(1, 2\theta)} = (1, \theta)^2.$$



Squares: $\frac{n}{2}$th roots.

Quiz

Which are the same as 1?

Quiz

Which are the same as 1?

(A) $(1)^2$
(B) $(-1)^2$
(C) -1
(D) $e^{2\pi i}$
(E) $(e^{\pi i})^2$

Quiz

Which are the same as 1?

(A) $(1)^2$
(B) $(-1)^2$
(C) -1
(D) $e^{2\pi i}$
(E) $(e^{\pi i})^2$

Which are the same as -1?

Quiz

Which are the same as 1?

(A) $(1)^2$
(B) $(-1)^2$
(C) -1
(D) $e^{2\pi i}$
(E) $(e^{\pi i})^2$

Which are the same as -1?

(A) $(-1)^2$
(B) $(e^{3\pi i/2})^2$
(C) $(e^{\pi i/2})^2$
(D) $(e^{\pi i})^2$

Quiz

Which are the same as 1?

(A) $(1)^2$
(B) $(-1)^2$
(C) -1
(D) $e^{2\pi i}$
(E) $(e^{\pi i})^2$

Which are the same as -1?

(A) $(-1)^2$
(B) $(e^{3\pi i/2})^2$
(C) $(e^{\pi i/2})^2$
(D) $(e^{\pi i})^2$

Note: $e^{\pi i} = -1$.

Quiz

Which are the same as 1?

(A) $(1)^2$
(B) $(-1)^2$
(C) -1
(D) $e^{2\pi i}$
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(A) $(-1)^2$
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(C) $(e^{\pi i/2})^2$
(D) $(e^{\pi i})^2$

Note: $e^{\pi i} = -1$. (B) $(e^{3\pi i/2})^2 = e^{3\pi i} = e^{\pi i}$

Quiz

Which are the same as 1?

(A) $(1)^2$
(B) $(-1)^2$
(C) -1
(D) $e^{2\pi i}$
(E) $(e^{\pi i})^2$

Which are the same as -1?

(A) $(-1)^2$
(B) $(e^{3\pi i/2})^2$
(C) $(e^{\pi i/2})^2$
(D) $(e^{\pi i})^2$

Note: $e^{\pi i} = -1$. (B) $(e^{3\pi i/2})^2 = e^{3\pi i} = e^{\pi i}$ (D) $(e^{\pi i/2})^2 = e^{\pi i}$.

Quiz

Which are the same as 1?

(A) $(1)^2$
(B) $(-1)^2$
(C) -1
(D) $e^{2\pi i}$
(E) $(e^{\pi i})^2$

Which are the same as -1?

(A) $(-1)^2$
(B) $(e^{3\pi i/2})^2$
(C) $(e^{\pi i/2})^2$
(D) $(e^{\pi i})^2$

Note: $e^{\pi i} = -1$. (B) $(e^{3\pi i/2})^2 = e^{3\pi i} = e^{\pi i}$ (D) $(e^{\pi i/2})^2 = e^{\pi i}$.

Which are 4th roots of unity? (Hint: take the 4th power.)

Quiz

Which are the same as 1?

(A) $(1)^2$
(B) $(-1)^2$
(C) -1
(D) $e^{2\pi i}$
(E) $(e^{\pi i})^2$

Which are the same as -1?

(A) $(-1)^2$
(B) $(e^{3\pi i/2})^2$
(C) $(e^{\pi i/2})^2$
(D) $(e^{\pi i})^2$

Note: $e^{\pi i} = -1$. (B) $(e^{3\pi i/2})^2 = e^{3\pi i} = e^{\pi i}$ (D) $(e^{\pi i/2})^2 = e^{\pi i}$.

Which are 4th roots of unity? (Hint: take the 4th power.)

(A) $e^{\pi i/2}$
(B) $e^{\pi i}$
(C) $e^{\pi i/3}$
(D) $e^{3\pi i/2}$

Quiz

Which are the same as 1?

(A) $(1)^2$
(B) $(-1)^2$
(C) -1
(D) $e^{2\pi i}$
(E) $(e^{\pi i})^2$

Which are the same as -1?

(A) $(-1)^2$
(B) $(e^{3\pi i/2})^2$
(C) $(e^{\pi i/2})^2$
(D) $(e^{\pi i})^2$

Note: $e^{\pi i} = -1$. (B) $(e^{3\pi i/2})^2 = e^{3\pi i} = e^{\pi i}$ (D) $(e^{\pi i/2})^2 = e^{\pi i}$.

Which are 4th roots of unity? (Hint: take the 4th power.)

(A) $e^{\pi i/2}$
(B) $e^{\pi i}$
(C) $e^{\pi i/3}$
(D) $e^{3\pi i/2}$

(A), (B)

Quiz

Which are the same as 1?

(A) $(1)^2$
(B) $(-1)^2$
(C) -1
(D) $e^{2\pi i}$
(E) $(e^{\pi i})^2$

Which are the same as -1?

(A) $(-1)^2$
(B) $(e^{3\pi i/2})^2$
(C) $(e^{\pi i/2})^2$
(D) $(e^{\pi i})^2$

Note: $e^{\pi i} = -1$. (B) $(e^{3\pi i/2})^2 = e^{3\pi i} = e^{\pi i}$ (D) $(e^{\pi i/2})^2 = e^{\pi i}$.

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(A), (B) and (D).

The FFT!

Defn: $\omega = (1, \frac{2\pi}{n}) = e^{\frac{2\pi i}{n}}$, nth root of unity.

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Evaluate $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_{n-1} x^{n-1}$

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Recursively compute A_e and A_o on $\frac{n}{2}$ roots of unity:

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Recursively compute A_e and A_o on $\frac{n}{2}$ roots of unity:
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$\omega^2, \omega^4, \omega^6, \ldots, \omega^n$.

For each $j \leq \frac{n}{2}$.

$$A(\omega^j) = A_e(\omega^{2i}) + \omega^j A_o(\omega^{2j})$$

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Runtime Recurrence:

The FFT!

Defn: $\omega = (1, \frac{2\pi}{n}) = e^{\frac{2\pi i}{n}}$, nth root of unity.
Pairs: ω^i and $\omega^{i+\frac{n}{2}} = \omega^i \omega^{\frac{n}{2}} = -\omega^i$. Common square!
Common Squares: are $\frac{n}{2}$ root of unity.

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Runtime Recurrence:
A_e and A_o are degree $\frac{n}{2}$, $\frac{n}{2}$ points in recursion.

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A_e and A_o are degree $\frac{n}{2}$, $\frac{n}{2}$ points in recursion.
$T(n) = 2T(\frac{n}{2}) + O(n)$

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Pairs: ω^i and $\omega^{i+\frac{n}{2}} = \omega^i \omega^{\frac{n}{2}} = -\omega^i$. Common square!
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Runtime Recurrence:
A_e and A_o are degree $\frac{n}{2}$, $\frac{n}{2}$ points in recursion.
$T(n) = 2T(\frac{n}{2}) + O(n) = O(n \log n)$!

Quiz 2: review

What is ω_n^n?

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What is ω_n^n? 1

Quiz 2: review

What is ω_n^n? 1

What is $(\omega_n)^{a+n}$?

Quiz 2: review

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What is $(\omega_n)^{a+n}$? ω_n^a.

What is $(\omega_n^{a+n/2})^2$?

Quiz 2: review

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Consider n points: $S_n = \{\omega_n, (\omega_n)^2, \ldots, \omega_n^n\}$.

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Consider n points: $S_n = \{\omega_n, (\omega_n)^2, \ldots, \omega_n^n\}$.

How many points in the set: $\{(\omega_n)^2, (\omega_n)^4, \ldots, \omega_n^{2n}\}$?

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$n/2$ points!!!

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FFT: Evaluate degree n polynomial on n points
by evaluating two degree $n/2$ polynomials on $n/2$ points!

Summary.

Polynomial Multiplication: $O(n^2)$.

Summary.

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In Point form: $O(n)$.

Summary.

Polynomial Multiplication: $O(n^2)$.

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Polynomial Evaluation: $O(n^2)$.

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Polynomial: $A(x) = A_e(x^2) + xA_o(x^2)$

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Evaluate on n points recursively.

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The number of leaves is n.

Summary.

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Consider n points: $S_n = \{\omega_n, (\omega_n)^2, \ldots, \omega_n^n\}$.

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Consider n points: $S_n = \{\omega_n, (\omega_n)^2, \ldots, \omega_n^n\}$.

Set of squares: $S_{n/2} = \{\omega_n^2), \omega_n)^4, \ldots, (\omega_n)^n, (\omega_n)^{n+2}, ...(\omega_n)^{2n}\}$.
Set of squares: $S_{n/2} = \{\omega_n^2), \omega_n)^4, \ldots, (\omega_n)^n\}$.
Only $n/2$ values here.

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Evaluate $A(x) = A_e(x^2) + xA_o(x^2)$.
Only need to evaluate A_e and A_o on $n/2$ points.
$T(n,n) = 2T(n/2,n/2) + O(n)$.
Or $T(n) = 2(n/2) + O(n) = O(n \log n)$