Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Counting inversions

This problem arises in the analysis of *rankings*. Consider comparing two rankings. One way is to label the elements (books, movies, etc.) from 1 to k according to one of the rankings, then order these labels according to the other ranking, and see how many pairs are "out of order".

We are given a sequence of k distinct numbers $n_1, \cdots, n_k$. We say that two indices $i < j$ form an inversion if $n_i > n_j$, that is if the two elements n_i and n_j are "out of order". Provide a divide and conquer algorithm to determine the number of inversions in the sequence $n_1, \cdots, n_k$ in time $\mathcal{O}(k \log k)$. Only an algorithm description and runtime analysis is needed.

Hint: Modify merge sort to count during merging.

2 Pareto Optimality

Given a set of points $P = \{(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)\}$, a point $(x_i, y_i) \in P$ is Pareto-optimal if there does not exist any $j \neq i$ such that such that $x_j > x_i$ and $y_j > y_i$. In other words, there is no point in P above and to the right of (x_i, y_i). Design a $O(n \log n)$-time divide-and-conquer algorithm that given P, outputs all Pareto-optimal points in P. Only an algorithm description and runtime analysis is needed.

Hint: Split the array by x-coordinate. Show that all points returned by one of the two recursive calls is Pareto-optimal, and that you can get rid of all non-Pareto-optimal points in the other recursive call in linear time.

3 Monotone matrices

A m-by-n matrix A is *monotone* if $n \geq m$, each row of A has no duplicate entries, and it has the following property: if the minimum of row i is located at column j_i, then $j_1 < j_2 < j_3 \ldots j_m$. For example, the following matrix is monotone (the minimum of each row is bolded):

$$\begin{bmatrix} \mathbf{1} & 3 & 4 & 6 & 5 & 2 \\ 7 & 3 & \mathbf{2} & 5 & 6 & 4 \\ 7 & 9 & 6 & 3 & 10 & \mathbf{0} \end{bmatrix}$$

Give an efficient (i.e., better than $O(mn)$-time) algorithm that finds the minimum in each row of an m-by-n monotone matrix A.

Give a 3-part solution. You do not need to write a formal recurrence relation in your runtime analysis; an informal summary of the runtime analysis such as "proof by picture" is fine.

4 FFT Intro

We will use ω_n to denote the first n-th root of unity $\omega_n = e^{2\pi i/n}$. The most important fact about roots of unity for our purposes is that the squares of the $2n$-th roots of unity *are* the n-th roots of unity.

Fast Fourier Transform! The *Fast Fourier Transform* $\text{FFT}(p, n)$ takes arguments n (an integer power of 2), and p (some vector $[p_0, p_1, \ldots, p_{n-1}]$).

Here, we describe how we can view FFT as a way to perform a specific matrix multiplication involving the DFT matrix. Note, however, that the FFT algorithm will not explicitly compute this matrix. We have written out the matrix below for convenience.

Treating p as a polynomial $P(x) = p_0 + p_1 x + \ldots + p_{n-1}x^{n-1}$, the FFT computes the value of $P(x)$ for all x that are n-th roots of unity by computing the result of the following matrix multiplication in $\mathcal{O}(n \log n)$ time:

$$\begin{bmatrix} P(1) \\ P(\omega_n) \\ P(\omega_n^2) \\ \vdots \\ P(\omega_n^{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n^1 & \omega_n^2 & \ldots & \omega_n^{(n-1)} \\ 1 & \omega_n^2 & \omega_n^4 & \ldots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{(n-1)} & \omega_n^{2(n-1)} & \ldots & \omega_n^{(n-1)(n-1)} \end{bmatrix} \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{n-1} \end{bmatrix}$$

If we let $E(x) = p_0 + p_2 x + \ldots p_{n-2}x^{n/2-1}$ and $O(x) = p_1 + p_3 x + \ldots p_{n-1}x^{n/2-1}$, then $P(x) = E(x^2) + xO(x^2)$, and then $FFT(p, n)$ can be expressed as a divide-and-conquer algorithm:

1. Compute $E' = \text{FFT}(E, n/2)$ and $O' = \text{FFT}(O, n/2)$.
2. For $i = 0 \ldots n-1$, assign $P(\omega_n^i) \leftarrow E((\omega_n^i)^2) + \omega_n^i O((\omega_n^i)^2)$

Also observe that:

$$\frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \ldots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-4} & \ldots & \omega_n^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \ldots & \omega_n^{-(n-1)(n-1)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n^{1} & \omega_n^{2} & \ldots & \omega_n^{(n-1)} \\ 1 & \omega_n^{2} & \omega_n^{4} & \ldots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{(n-1)} & \omega_n^{2(n-1)} & \ldots & \omega_n^{(n-1)(n-1)} \end{bmatrix}^{-1}$$

(You should verify this on your own!) And so given the values $P(1), P(\omega_n), P(\omega_n^2) \ldots$, we can compute P by finding the result of the following matrix multiplication in $O(n \log n)$ time:

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{n-1} \end{bmatrix} = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \ldots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-4} & \ldots & \omega_n^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \ldots & \omega_n^{-(n-1)(n-1)} \end{bmatrix} \cdot \begin{bmatrix} P(1) \\ P(\omega_n) \\ P(\omega_n^2) \\ \vdots \\ P(\omega_n^{n-1}) \end{bmatrix}$$

This can be done in $O(n \log n)$ time using a similar divide and conquer algorithm.

(a) Let $p = [p_0]$. What is $\text{FFT}(p, 1)$?

(b) Use the FFT algorithm to compute $\text{FFT}([1, 4], 2)$ and $\text{FFT}([3, 2], 2)$.

(c) Use your answers to the previous parts to compute $\text{FFT}([1, 3, 4, 2], 4)$.

(d) Describe how to multiply two polynomials $P(x), Q(x)$ in coefficient form of degree at most d.

(e) Use the algorithm from the previous part to multiply the two polynomials $P(x) = 1 + 2x$ and $Q(x) = 3 - x$ in coefficient form.