Prim(G, w)

$\forall u \in V$ cost$(u) = \infty$,
Pick any $u_0 \in V$
cost$(u_0) = 0$ prev$(u_0) = u_0$
$\forall v \in V$ insert key$(v$, cost$(v))$
while queue non empty
  v = Delete Min
  $\forall \{v, u\} \in E$
    if cost$(u) > w(v, u)$
      cost$(u) = w(v, u)$
      prev$(u) = v$
      DecreaseKey(u)
Output pre$[u]$ $\forall u \in V$

Claim: Interpreting pre(u) as the parent of u, Prim(G, w) outputs a MST rooted at u_0

Proof: Let Q_t be the queue after t steps, with $Q_0 = V$, and let $S_t = V \setminus S_t$.

The algorithm starts with

$$\text{cost}(u_0) = 0, \ \text{cost}(u) = \infty \ \ \forall u \neq u_0$$

so after the first round $S_1 = \{u_0\}$ and

$$\text{cost}(u) = W_{u_0 u}$$

$$\text{prev}(u) = u_0$$

For all $u \in Q_1$ s.th. $u_0 u$ is an edge from S_1 to $Q_1 = V \setminus S_1$. So when we call DeleteMin(Q_1) and set $v_1 =$ DeleteMin(Q_1)

$$w_{u_0 v_1} = \min_{\substack{u \in S_1 \\ v \in V \setminus S_1 \\ uv \in E}} w_{uv}$$

and the edge $u_0 v_1 = pre(v_1) v_1$ is correctly added to the MST in accordance with the cut property.

The following claim ensures that holds for all steps

Claim: Just before the Deletemin is executed for the t^{th} time the following two equalities hold for all $u \in Q_t$

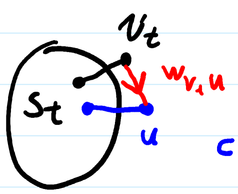
$$cost(u) = \min_{\substack{v \in S_t \\ vu \in E}} w_{vu} = w_{prev(u)\, u}$$

Proof by induction

Base case $t = 1$ ✓

$\boxed{t \to t+1}$

Let v_t = delete min Q_t



$$cost(u) = \min_{v \in S_t} w_{vu} = w_{pre(u)\, u}$$

at end of last round

Updating S_t gives $S_{t+1} = S_t \cup v_t$.

If $w_{v_t u} < cost(u)$ we correctly update both

$cost(u)$ to $w_{v_t u}$ and $prev(u) = v_t$, see figure.

Otherwise, no update is needed