Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Horn Formula Practice

(a) Find the variable assignment that solves the following horn formula:

$$(x \wedge z) \Rightarrow y, z \Rightarrow w, (y \wedge z) \Rightarrow x, \Rightarrow z, (\bar{z} \vee \bar{x}), (\bar{w} \vee \bar{y} \vee \bar{z})$$

(b) Show that any implication clause of the form $(x_i \wedge x_j \wedge \cdots) \Rightarrow \mathsf{True}$ is always satisfiable.
Hint: what disjunction clause is this equivalent to?

(c) Show that any implication clause of the form $\mathsf{False} \Rightarrow x_k$ is always satisfiable.

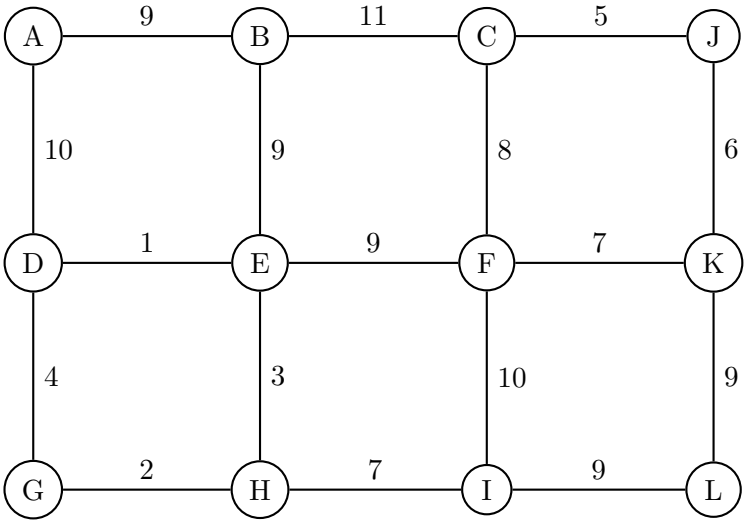
2 Huffman Proofs

(a) Prove that in the Huffman coding scheme, if some symbol occurs with frequency more than $\frac{2}{5}$, then there is guaranteed to be a codeword of length 1.

(b) Prove that in the Huffman coding scheme, if all symbols occur with frequency less than $\frac{1}{3}$, then there is guaranteed to be no codeword of length 1.

(c) Suppose that our alphabet consists of n symbols. What is the longest possible encoding of a single symbol under the Huffman code? What set of frequencies yields such an encoding?

3 MST Tutorial



(a) List the first **six** edges added by Prim's algorithm in the order in which they are added. Assume that Prim's algorithm starts at vertex A and breaks ties lexicographically.

(b) List the first **seven** edges added by Kruskal's algorithm in the order in which they are added. You may break ties in any way.

(c) Prim's algorithm is very similar to Dijkstra's in that a vertex is processed at each step which minimizes some cost function. These algorithms also produce similar outputs: the union of all shortest paths produced by a run of Dijkstra's algorithm forms a tree. However, the trees they produce aren't optimizing for the same thing. To see this, give an example of a graph for which different trees are produced by running Prim's algorithm and Dijkstra's algorithm. In other words, give a graph where there is a shortest path from a start vertex A using at least one edge that doesn't appear in any MST.

4 MST Potpourri

(a) Given an undirected graph $G = (V, E)$ and a set $E' \subset E$ briefly describe how to update Kruskal's algorithm to find the minimum spanning tree that includes all edges from E'.

(b) Suppose we want to find the minimum cost set of edges that suffices to connect a given weighted graph $G = (V, E)$; if the weights are non-negative then we know that the optimum will be a MST. What about the case when the weights are allowed to be negative? Does it have to be a tree if the weights are allowed to be negative? If not, how would you find this minimum-cost connected subgraph?

(c) Describe an algorithm to find a maximum spanning tree of a given graph.

5 Penguin Phone Plans

The PNPenguins' ice-cream selling business has really taken off, and they would really like to share the great news with all their n friends. The PNPenguins have friends who live internationally, so having to call each friend individually could be expensive, as the price of phone calls increases with distance. Luckily, some of their n friends also know each other, so they can call one another and share the news (gossip!). For two people i and j, the PNPenguins know if i and j are friends and know the distance $d(i, j)$ between them.

(a) Help the PNPenguins design an algorithm to determine how phone calls should be made to minimize the total phone bill when sharing the news with all their friends. Assume that the group of PNPenguins call their friends together.

(b) Seeing the pattern of phone calls, P-Mobile (a telecommunications company) has changed their phone plan so that the price of calls within a certain distance are halved, and calls above a certain distance are doubled. Does the solution returned from part (a) change? Briefly justify.

(c) Following their success in the Antarctic, PNPenguins have decided to expand their business into the Arctic, stationing at least one penguin in each of the m regions of the Arctic. Hearing this, P-Mobile has decided to expand into the Arctic as well. P-Mobile knows that the PNPenguins have to make phone calls to coordinate their ice-cream selling business across all regions of the Arctic. P-Mobile has decided that calls within a region should be free, and calls between different regions should be priced proportionally to the distance between the regions. They have also decided to only allow calls between certain specific regions. Design an algorithm for P-Mobile to find a connection between regions that maximizes revenue from PNPenguin calls.

6 Updating a MST

You are given a graph $G = (V, E)$ with positive edge weights, and a minimum spanning tree $T = (V, E')$ with respect to these weights; you may assume G and T are given as adjacency lists. Now suppose the weight of a particular edge $e \in E$ is modified from $w(e)$ to a new value $\hat{w}(e)$. You wish to quickly update the minimum spanning tree T to reflect this change, without recomputing the entire tree from scratch.

There are four cases. In each, give a description of an algorithm for updating T, a proof of correctness, and a runtime analysis for the algorithm. Note that for some of the cases these may be quite brief. For simplicity, you may assume that no two edges have the same weight (this applies to both w and $\hat{w}$).

(a) $e \in E'$ and $\hat{w}(e) < w(e)$

(b) $e \notin E'$ and $\hat{w}(e) < w(e)$

(c) $e \in E'$ and $\hat{w}(e) > w(e)$

(d) $e \notin E'$ and $\hat{w}(e) > w(e)$