CS 70 Discrete Mathematics and Probability Theory
Spring 2023 Babak Ayazifar and Satish Rao

HW 12

Due: Saturday, 4/15, 4:00 PM
Grace period until Saturday, 4/15, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Double-Check Your Intuition Again

Note 16

(a) You roll a fair six-sided die and record the result X. You roll the die again and record the result Y.

(i) What is $\text{cov}(X+Y, X-Y)$?

(ii) Prove that $X+Y$ and $X-Y$ are not independent.

For each of the problems below, if you think the answer is "yes" then provide a proof. If you think the answer is "no", then provide a counterexample.

(b) If X is a random variable and $\text{Var}(X) = 0$, then must X be a constant?

(c) If X is a random variable and c is a constant, then is $\text{Var}(cX) = c\,\text{Var}(X)$?

(d) If A and B are random variables with nonzero standard deviations and $\text{Corr}(A,B) = 0$, then are A and B independent?

(e) If X and Y are not necessarily independent random variables, but $\text{Corr}(X,Y) = 0$, and X and Y have nonzero standard deviations, then is $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$?

(f) If X and Y are random variables then is $\mathbb{E}[\max(X,Y)\min(X,Y)] = \mathbb{E}[XY]$?

(g) If X and Y are independent random variables with nonzero standard deviations, then is

$$\text{Corr}(\max(X,Y), \min(X,Y)) = \text{Corr}(X,Y)?$$

2 Unreliable Servers

Note 19

A Google competitor owns a warehouse consisting of a very large number of servers (a server farm). On any given day, each server in the farm is equally likely to go down or to stay online, independently of all other servers, and independently of what happens on any number of other days. On average, 4 servers go down in the cluster per day.

(a) What is an appropriate distribution to model the number of servers that crash on any given day for a certain cluster? (Give the name and parameter(s) of the distribution.)

(b) Compute the expected value and variance of the number of crashed servers on a given day for a certain cluster.

(c) Compute the probability that strictly less than 3 servers crashed on a given day for a certain cluster.

(d) Compute the probability that at least 3 servers crashed on a given day for a certain cluster.

(e) Compute the probability that exactly 6 servers crashed over a given two-day period for a certain cluster.

3 Geometric and Poisson

Note 19

Let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Poisson}(\lambda)$ be independent random variables. Compute $\mathbb{P}[X > Y]$. Your final answer should not have summations.

Hint: Use the total probability rule.

4 Coupon Collector Variance

Note 19

It's that time of the year again—Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let X be the number of visits you have to make before you can redeem the grand prize. Show that $\text{Var}(X) = n^2 \left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}[X]$.

5 Probabilistically Buying Probability Books

Note 20

Chuck will go shopping for probability books for K hours. Here, K is a random variable and is equally likely to be 1, 2, or 3. The number of books N that he buys is random and depends on how long he shops. We are told that

$$\mathbb{P}[N = n \mid K = k] = \begin{cases} \frac{c}{k} & \text{for } n = 1, \ldots, k \\ 0 & \text{otherwise} \end{cases}$$

for some constant c.

(a) Compute c.

(b) Find the joint distribution of K and N.

(c) Find the marginal distribution of N.

(d) Find the conditional distribution of K given that $N = 1$.

(e) We are now told that he bought at least 1 but no more than 2 books. Find the conditional mean and variance of K, given this piece of information. (*Hint*: For a random variable X and event A, we have $\text{Var}(X \mid A) = \mathbb{E}[X^2 \mid A] - \mathbb{E}[X \mid A]^2$.)

(f) The cost of each book is a random variable with mean 3. What is the expectation of his total expenditure? *Hint:* Condition on events $N = 1$, $N = 2$, $N = 3$ and use total expectation.

6 Dice Games

Note 20

(a) Alice and Bob are playing a game. Alice picks a random integer X between 0 and 100 inclusive, where each value is equally likely to be chosen. Bob then picks a random integer Y between 0 and X inclusive. What is $\mathbb{E}[Y]$?

(b) Alice rolls a die until she gets a 1. Let X be the number of total rolls she makes (including the last one), and let Y be the number of rolls on which she gets an even number. Compute $\mathbb{E}[Y \mid X = x]$, and use it to calculate $\mathbb{E}[Y]$.

(c) Bob plays a game in which he starts off with one die. At each time step, he rolls all the dice he has. Then, for each die, if it comes up as an odd number, he puts that die back, and adds a number of dice equal to the number displayed to his collection. (For example, if he rolls a one on the first time step, he puts that die back along with an extra die.) However, if it comes up as an even number, he removes that die from his collection.

What is the expected number of dice Bob will have after n time steps?