Finish: Error Correction scheme.

Message “is” points on $P(x)$. (degree $n-1$, $n+2k$ points.)

Channel: Send $P(i)$, receive $R(i)$.

Errors are wrong values at $\leq k$ points.Error: $P(i) \neq R(i)$.

Error locator polynomial:
$E(x) = (x-e_1)\cdot(x-e_k) = x^k + b_{k-1}x^{k-1} + \cdots + b_0$.

Find: $Q(x) = E(x)P(x) = a_{n+k-1}x^{n+k-1} + \cdots + a_0$ and $E(x)$.

$P(x) = Q(x)/E(x)$.

Solving for $Q(x)$ and $E(x)$...and $P(x)$

For all points $1,\ldots,i,n+2k = m$,

$$Q(i) = R(i)E(i) \pmod{p}$$

Gives $n+2k$ linear equations.

$$\begin{aligned}
a_{n+k-1} + \ldots a_0 &\equiv R(1)(1 + b_{k-1}\cdots b_0) \pmod{p}\\
a_{n+k-1}(2)^{n+k-1} + \ldots a_0 &\equiv R(2)((2)^k + b_{k-1}(2)^{k-1}\cdots b_0) \pmod{p}\\
&\vdots\\
a_{n+k-1}(m)^{n+k-1} + \ldots a_0 &\equiv R(m)((m)^k + b_{k-1}(m)^{k-1}\cdots b_0) \pmod{p}
\end{aligned}$$

..and $n+2k$ unknown coefficients of $Q(x)$ and $E(x)$!

Solve for coefficients of $Q(x)$ and $E(x)$.

Find $P(x) = Q(x)/E(x)$.

Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

$Q(x) = E(x)P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$E(x) = x - b_0$

$Q(i) = R(i)E(i)$.

$$\begin{aligned}
a_3 + a_2 + a_1 + a_0 &\equiv 3(1-b_0) \pmod{7}\\
a_3 + 4a_2 + 2a_1 + a_0 &\equiv 1(2-b_0) \pmod{7}\\
6a_3 + 2a_2 + 3a_1 + a_0 &\equiv 6(3-b_0) \pmod{7}\\
a_3 + 2a_2 + 4a_1 + a_0 &\equiv 0(4-b_0) \pmod{7}\\
6a_3 + 4a_2 + 5a_1 + a_0 &\equiv 3(5-b_0) \pmod{7}
\end{aligned}$$

$a_3 = 1$, $a_2 = 6$, $a_1 = 6$, $a_0 = 5$ and $b_0 = 2$.

$Q(x) = x^3 + 6x^2 + 6x + 5$.

$E(x) = x - 2$.

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5$.
$E(x) = x - 2$.

```
                 1 x^2 + 1 x + 1
          -----------------
      x - 2 ) x^3  + 6 x^2 + 6 x + 5
              x^3  - 2 x^2
              ----------
                 1 x^2 + 6 x + 5
                 1 x^2 - 2 x
                 ---------------
                           x + 5
                           x - 2
                           -----
                               0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6$.

What is $\frac{x-2}{x-2}$? 1
Except at $x = 2$? Hole there?

Error Correction: Berlekamp-Welsh

Message: $m_1,\ldots,m_n$.
Sender:

1. Form degree $n-1$ polynomial $P(x)$ where $P(i) = m_i$.
2. Send $P(1),\ldots,P(n+2k)$.

Receiver:

1. Receive $R(1),\ldots,R(n+2k)$.
2. Solve $n+2k$ equations, $Q(i) = E(i)R(i)$ to find $Q(x) = E(x)P(x)$ and $E(x)$.
3. Compute $P(x) = Q(x)/E(x)$.
4. Compute $P(1),\ldots,P(n)$.

Check your undersanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Factor? Sure.
Check all values? Sure.

Efficiency? Sure. Only $n+2k$ values.
See where it is 0.

Hmmm...

Is there one and only one $P(x)$ from Berlekamp-Welsh procedure?

Existence: there is a $P(x)$ and $E(x)$ that satisfy equations.

Unique solution for $P(x)$

Uniqueness: any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x). \qquad (1)$$

Proof:
We claim

$$Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x. \qquad (2)$$

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$
and agree on $n+2k$ points
$E(x)$ and $E'(x)$ have at most k zeros each.
Can cross divide at n points.
$\implies \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)}$ equal on n points.
Both degree $\leq n-1 \implies$ Same polynomial! □

Last bit.

Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n+2k$ values of x.

Proof: Construction implies that

$$Q(i) = R(i)E(i)$$
$$Q'(i) = R(i)E'(i)$$

for $i \in \{1,\dots n+2k\}$.

If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$.
$\implies Q(i)E'(i) = Q'(i)E(i)$ holds when $E(i)$ or $E'(i)$ are zero.

When $E'(i)$ and $E(i)$ are not zero

$$\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).$$

Cross multiplying gives equality in fact for these points. □

Points to polynomials, have to deal with zeros!

Example: dealing with $\frac{x-2}{x-2}$ at $x = 2$.

Yaay!!

Berlekamp-Welsh algorithm decodes correctly when k errors!

Poll

Say you sent a message of length 4, encoded as $P(x)$ where one sends packets $P(1),\dots P(8)$.

You recieve packets $R(1),\dots R(8)$.

Packets 1 and 4 are corrupted.

(A) $R(1) \neq P(1)$
(B) The degree of $P(x)E(x) = 3+2 = 5$.
(C) The degree of $E(x)$ is 2.
(D) The number of coefficients of $P(x)$ is 4.
(E) The number of coefficients of $P(x)Q(x)$ is 6.

(E) is false.

(A) $E(x) = (x-1)(x-4)$
(B) The number of coefficents in $E(x)$ is 2.
(C) The number of unknown coefficents in $E(x)$ is 2.
(D) $E(x) = (x-1)(x-2)$
(E) $R(4) \neq P(4)$
(F) The degree of $R(x)$ is 5.

(A), (C), (E). (F) doesn't type check!

Summary. Error Correction.

Communicate n packets, with k erasures.

How many packets? $n+k$
How to encode? With polynomial, $P(x)$.
Of degree? $n-1$
Recover? Reconstruct $P(x)$ with any n points!

Communicate n packets, with k errors.

How many packets? $n+2k$
Why?
k changes to make diff. messages overlap
How to encode? With polynomial, $P(x)$. Of degree? $n-1$.
Recover?
Reconstruct error polynomial, $E(X)$, and $P(x)$!
Nonlinear equations.
Reconstruct $E(x)$ and $Q(x) = E(x)P(x)$. Linear Equations.
Polynomial division! $P(x) = Q(x)/E(x)$!

Reed-Solomon codes. Welsh-Berlekamp Decoding. Perfection!

Cool.

Really Cool!

Poll: How big is infinity?

Mark what's true.
(A) There are more real numbers than natural numbers.
(B) There are more rational numbers than natural numbers.
(C) There are more integers than natural numbers.
(D) pairs of natural numbers $>>$ natural numbers.

Same Size. Poll.

Two sets are the same size?

(A) Bijection between the sets.
(B) Count the objects and get the same number. same size.
(C) Counting to infinity is hard.

(A), (B).
(C)?

Countable.

How to count?

0, 1, 2, 3, $\ldots$

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,
output only if $x \in T$.

Implications:
Z^+ is countable.
It is infinite since the list goes on.
There is a bijection with the natural numbers.
So it is countably infinite.

All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$.
ϕ is empty string.

For any string, it appears at some position in the list.
If n bits, it will appear before position 2^{n+1}.

Should be careful here.

$B = \{\phi; , 0, 00, 000, 0000, ...\}$
Never get to 1.

More fractions?

Enumerate the rational numbers in order...

$0,\ldots,1/2,..$

Where is $1/2$ in list?

After $1/3$, which is after $1/4$, which is after $1/5$...

A thing about fractions:
any two fractions has another fraction between it.

Can't even get to "next" fraction!

Can't list in "order".

Pairs of natural numbers.

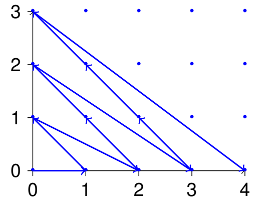
Consider pairs of natural numbers: $N \times N$
E.g.: $(1,2)$, $(100,30)$, etc.

For finite sets S_1 and S_2,
then $S_1 \times S_2$
has size $|S_1| \times |S_2|$.

So, $N \times N$ is countably infinite squared ???

Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots$...



The pair (a,b), is in first $\approx (a+b+1)(a+b)/2$ elements of list!
(i.e., "triangle").

Countably infinite.

Same size as the natural numbers!!

Poll.

Enumeration to get bijection with naturals?

(A) Integers: First all negatives, then positives.
(B) Integers: By absolute value, break ties however.
(C) Pairs of naturals: by sum of values, break ties however.
(D) Pairs of naturals: by value of first element.
(E) Pairs of integers: by sum of values, break ties.
(F) Pairs of integers: by sum of absolute values, break ties.

(B),(C), (F).

Rationals?

Positive rational number.
Lowest terms: a/b
$a,b \in N$
with $gcd(a,b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative ??? No!

Repeatedly and alternatively take one from each list.
Interleave Streams in 61A

The rationals are countably infinite.

Real numbers..

Real numbers are same size as integers?

The reals.

Are the set of reals countable?

Lets consider the reals $[0,1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$
.632120558... $1-1/e$
.345212312... Some real number

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .77677...

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset $[0,1]$ is not countable!!

All reals?

Subset $[0,1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.

If reals are countable then so is $[0,1]$.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.
4. Show that t is different from all elements in the list
   $\implies$ t is not in the list.
5. Show that t is in S.
6. Contradiction.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.
otherwise $i \notin D$.

D is different from ith set in L for every i.
$\implies$ D is not in the listing.

D is a subset of N.

L does not contain all subsets of N.

Contradiction.

Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S, is the **powerset** of N.)

Poll: diagonalization Proof.

Mark parts of proof.

(A) Integers are larger than naturals cuz obviously.
(B) Integers are countable cuz, interleaving bijection.
(C) Reals are uncountable cuz obviously!
(D) Reals can't be in a list: diagonal number not on list.
(E) Powerset in list: diagonal set not in list.

(B), (C)?, (D), (E)

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

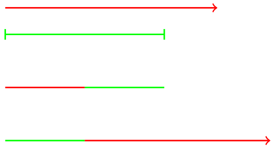
First of Hilbert's problems!

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f: R^+ \rightarrow [0,1]$.

$$f(x) = \begin{cases} x+\frac{1}{2} & 0 \le x \le 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$



One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0,1/2]$ a division $\implies f(x) \neq f(y)$.
If one is in $[0,1/2]$ and one isn't, different ranges $\implies f(x) \neq f(y)$.
Bijection!

$[0,1]$ is same cardinality as nonnegative reals!

Rao is freaked out.

Are real numbers even real?

Almost all real numbers can't be described.

π?

The ratio of the perimeter of a circle to its diameter.

e? Transendental number.
$\lim_{n\to\infty}(1+1/n)^n$.

$\sqrt{2}$? Algebraic number.
The solutions of

$$x^2 = 2$$

.

Really, rationals seem fine for calculus.

$$\lim_{n\to\infty}\sum_{i=0}^{n}\frac{(b-a)}{n}f(x_i),$$

where $x_i = \frac{a+i\times(b-a)/n}{.}$

So why real numbers?

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.

Resolution of hypothesis?

Gödel. 1940.
Can't use math!
If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Who shaves the barber?

Self reference.

Can a program refer to a program?

Can a program refer to itself?

Uh oh....

The Barber!

The barber shaves every person who does not shave themselves.

(A) Barber not Mark. Barber shaves Mark.
(B) Mark shaves the Barber.
(C) Barber doesn't shave themself.
(D) Barber shaves themself.

Its all true. It's all a problem.

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.

Recall: powerset of the naturals is not countable.

Resolution of hypothesis?

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Changing Axioms?

Goedel:
Any set of axioms is either
inconsistent (can prove false statements) or
incomplete (true statements cannot be proven.)

Concrete example:
Continuum hypothesis: "no cardinatity between reals and naturals."
Continuum hypothesis not disprovable in ZFC
(Goedel 1940.)

Continuum hypothesis not provable.
(Cohen 1963: only Fields medal in logic)

BTW:
Cantor ..bipolar disorder..
Goedel ..starved himself out of fear of being poisoned..
Russell .. was fine.....but for ...two schizophrenic children..
Dangerous work?

See Logicomix by Doxiaidis, Papadimitriou (was professor here), Papadatos, Di Donna.