Recap

- Discussed sets, set operations, set equality proofs
- Discussed functions (injections, surjections, bijections)
- Found bijection $f: \mathbb{N} \to 2\mathbb{N}$

But !

Today: Define cardinality for infinite sets.

Infinite Cardinality

Let's recall some stuff from last time.

Def: Let $f: X \to Y$ be a function.

- f is an injection if for all $x_1, x_2 \in X$,

- f is a surjection if for all $y \in Y$,

- f is a bijection if

Theorem: If X, Y are finite, there is a $f: X \to Y$ iff

Go back to bijection $f: \mathbb{N} \to 2\mathbb{N}$.

Moral: Can't assign to infinite sets!

Trick: Define for infinite sets using .

Def: If X, Y are sets, we write " " if there is a .

Subtle point: For finite sets, $|X| = |Y|$ already had a meaning, and the above was a property. Now, we extend the meaning of $|X| = |Y|$ to infinite sets using the property as a definition.

Warning: $|X|$ has no meaning if X is infinite. We only put

We do the same for $|X| \leq |Y|$.

Def: If X, Y are sets, we write if there is an .

$|X|=|Y|$ and $|X|\le|Y|$ is just notation.

But the notation suggests certain properties.

Property	Proof
$\lvert X\rvert=\lvert Y\rvert \Rightarrow \lvert X\rvert\le\lvert Y\rvert$	
$\lvert X\rvert=\lvert X\rvert$	
$\lvert X\rvert=\lvert Y\rvert \Rightarrow \lvert Y\rvert=\lvert X\rvert$	
$\lvert X\rvert=\lvert Y\rvert,\ \lvert Y\rvert=\lvert Z\rvert \Rightarrow \lvert X\rvert=\lvert Z\rvert$	

So, $|X|=|Y|$ has many expected properties of equality!

Countable Sets

Idea: $\mathbb{N}=\{0,1,2,\ldots\}$ is the "simplest" infinite set.

So, look for infinite sets same size as $\mathbb{N}$.

Def: • X is ____________ if $|X|=|\mathbb{N}|$.

• X is ________ if it is countably infinite or finite.

Why countable? Say X is countably infinite.

Then, there is a bijection $f: \mathbb{N} \to X$.

We can use f to ___ elements of X:

$\in X$

Since f is bijective, each $x \in X$ ____________

____________ .

Conversely: Listing elements of X gives a bijection $X \to \mathbb{N}$. How?

List: $x_0, x_1, x_2, \dots \in X$

f: ↓ ↓ ↓ $\longrightarrow$ If the list has each element of X exactly once,

Takeaway: $\Longleftrightarrow$

Ex: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is countable.

Proof: We can list $\mathbb{Z}$:

The list gives a bijection $\square$

Warning: Hidden list rules.

Consider listing $\mathbb{Z}$:

Q: What position does -1 have? A:

is <u>not</u> in $\mathbb{N}$, so this list does <u>not</u> give a bijection $\mathbb{Z} \to \mathbb{N}$.

Moral: When listing all elements must have ___ position.

Prop: Say $A_1, \dots, A_n$ are countably infinite sets with no overlap. Then is countably infinite.

Proof: For each i, A_i is countably infinite.

So,

Consider writing out lists.

$$\begin{array}{llll}
A_1: & a_{1,1} & a_{1,2} & a_{1,3} \cdots \\
A_2: & a_{2,1} & a_{2,2} & a_{2,3} \cdots \\
A_3: & a_{3,1} & a_{3,2} & a_{3,3} \cdots \\
\vdots & & & \\
A_n: & a_{n,1} & a_{n,2} & a_{n,3} \cdots
\end{array}$$

By zig-zagging through the lists like in blue, we get a list of elements of $\bigcup_{i=1}^{n} A_i$. Done! $\square$

Ex: $\mathbb{Q} = \{ \frac{a}{b} \mid a,b \in \mathbb{Z}, b \neq 0 \}$ is countable.

For simplicity, let's do $\mathbb{Q}^+$, the

To list positive rationals, make a grid:

$$\begin{array}{cccccc} 1/1 & 1/2 & 1/3 & 1/4 & \cdots \\ 2/1 & 2/2 & 2/3 & 2/4 & \\ 3/1 & 3/2 & 3/3 & & \\ 4/1 & 4/2 & & & \\ \vdots & & & & \end{array}$$

Problem! Our list has repeats!

For instance: $1/1 = 1$ and $2/2 = 1$.

So, the "function" $\mathbb{Q}^+ \to \mathbb{N}$ we get maps $1 \to 0$ and $1 \to 4$

Not even a real "well-defined" function!

"Fix": Map each element of $\mathbb{Q}^+$ to its position in list, e.g. ,

, , , ,...

Problem:

But this is an injection $\mathbb{Q}^+ \to \mathbb{N}$. So, we showed .

What about the other way? Consider $f: \mathbb{N} \to \mathbb{Q}^+$ given by . This is an injection!

So, !

Can we conclude ? Not right away!

Theorem (Cantor-Schöder-Bernstein): If there exist injections $X \to Y$ and $Y \to X$ then there is a bijection $X \to Y$.

(i.e. and $\Rightarrow$)

Proof: See typed notes. Harder than most 70 proofs, but understandable. $\square$

Using CSB: is countable!

Prop: $\mathbb{Q}$ is countable.

Proof: Can show $\mathbb{Q}^{\leq 0}$ is countable using same method as $\mathbb{Q}^{+}$.

Then, $\mathbb{Q} = \mathbb{Q}^{+} \cup \mathbb{Q}^{\leq 0}$ is countable by our earlier result. $\square$

Moral: If we want to show $|X| = |Y|$, sometimes finding two injections is easier.

Uncountable Sets

Let's consider the set of reals $\mathbb{R}$.

We will view $\mathbb{R}$ as the set of all numbers with an infinite decimal expansion.

$\frac{1}{2} =$

$\frac{4}{3} =$

$\pi =$

Theorem: $\mathbb{R}$ is uncountable (no bijection $\mathbb{N} \to \mathbb{R}$).

Proof: For contradiction, say $\mathbb{R}$ is countable.

Then, we can list its elements

In particular, can list elements of

Consider any list of $(0,1)$. For instance,

$$\begin{aligned}
&0.18442\cdots\\
&0.30664\cdots\\
&0.01847\cdots\\
&0.01101\cdots
\end{aligned}$$

Make a new number: Take circled digits and change digits to , change to .

So, get ________.

Notice: This new number is different than num in decimal, num in decimal, ...

So, it is ____ from the list!

Contradiction! $\square$

This is called a ____________.

Countability Summary

Prove countability: • List elements
• Find bijection to countable set

Prove uncountability: • Diagonalization
• Find bijection to uncountable set

Uncountability heuristic? 1. Set is infinite
2. Each element is infinitely complex

Final Ex: $\{f : \mathbb{N} \to \mathbb{N}\}$ is uncountable.

Proof: By diagonalization. Suppose we list all functions.

f_0 : . . .

f_1 : . . . .

⋮

Make a new function f so that

, , . . .

Missing from our list! $\square$

Computability

What is a computer program?

A piece of finite text that can be encoded in binary. So...

A computer program is a

The set of finite binary strings is ______ ( ) so set of computer programs is ______.

What functions might we want to implement as computer programs?

The input/output of a computer program can be viewed as a

There is a ______ set of possible inputs/outputs for a program!

So, $|\{f : \{\text{inputs}\} \to \{\text{outputs}\}\}| = |\qquad|$

Upshot: The set of functions which we might want to implement as programs is , but the set of programs is !

Some functions

We call such functions ________.

Can we find one?

Halting Problem

Consider the following function:

$$\text{TestHalt}(P, x) = \begin{cases} 1, \\ 0, \end{cases}$$

Idea: Checking if P(x) halts is hard... How do we know if it is stuck in a loop or about to finish?

Theorem (Turing): TestHalt is uncomputable.

Proof: Suppose we have written a computer program which implements TestHalt.
Consider defining another function.

```
def Turing(P):
    if TestHalt(P, P) == 1 : #
        enter an infinite loop
    else: #
        return
```

Consider Turing(Turing).

1. If Turing(Turing) halts, then
   TestHalt(Turing, Turing) == .
   But this means Turing(Turing) !
   Makes no sense.

2. If Turing(Turing) loops forever,
   TestHalt(Turing, Turing) == .
   But this means Turing(Turing) .
   Makes no sense.

So, Turing(Turing) cannot halt or loop forever. Contradiction! $\square$

We found an uncomputable function!

But the proof was weird...

Good news: We can use to show other things are uncomputable!

Ex: $\text{HaltsOnZero}(P) = \begin{cases} 1, & P(0) \\ 0, & P(0) \end{cases}$

binary rep of program

Prop: HaltsOnZero is uncomputable.

Strategy: Assume we implemented ,
and show we can implement .
We know implementing TestHalt is impossible,
so we get a contradiction!

This is called a ____ from
____ to ____.

Proof: For contradiction, suppose we implemented HaltsOnZero. We implement TestHalt.

```
def TestHalt(P, x):
    def Helper(n): #



    return HaltsOnZero(        )
```

If P(x) halts, Helper(0) , so HaltsOnZero(Helper) = . So, TestHalt(P, x) returns .

If $P(x)$ loops, Helper(0) , so
HaltsOnZero(Helper) = . So, TestHalt(P,x)
returns .
We implemented TestHalt, a contradiction! $\square$