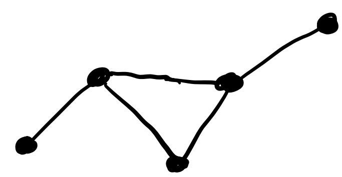
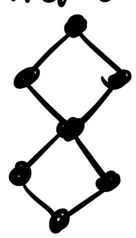
Recap

Graphs: $G=(V,E)$



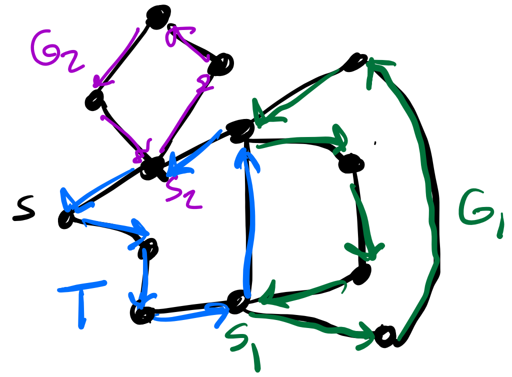
- Handshaking Lemma: $\sum_{v\in V} \deg v = 2|E|$
  ↳ # edges touching v
- Types: Complete, bipartite, trees
  - Complete ↓ all possible edges
  - bipartite ↳ vertices in 2 groups, no edges within group
  - trees ↳ connected + no cycles
- Traversals: Cycle, Eulerian Tour



  - Cycle ↓ loop w/ no repeated edges/vertices
  - Eulerian Tour ↳ loop which uses <u>all</u> edges exactly once

Theorem: Let $G=(V,E)$ be a graph.
G has an Eulerian tour iff $\deg v$ is even for all $v\in V$ <u>and</u> G is connected (except for isolated vertices).

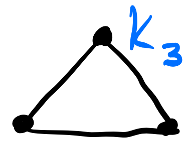
```
def Euler(G, s):
    T = FindTour(G, s)
    G_1, ..., G_k = Connected comps of G after
                    removing edges in T
    s_i = first vertex in T intersecting G_i
    return Splice(T, Euler(G_1, s_1), ..., Euler(G_k, s_k))
```



Ex: Conditions for Eulerian tour

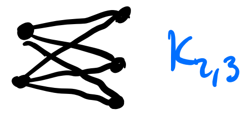
1. Complete graph K_n : E.T. $\iff$ n

   Each vertex has degree



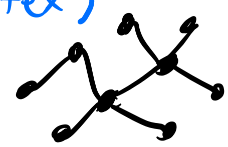
2. Complete bipartite graph $K_{n,m}$: E.T. $\iff$ n,m

   All vertices have degree or



3. Tree: E.T. $\iff$

   All other trees have a (vertex)



Hypercubes

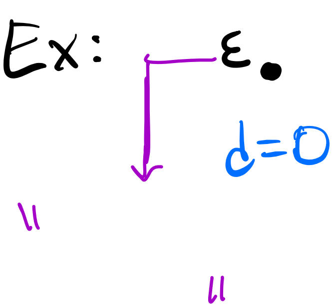
- More edges than trees, fewer than complete
- Like complete graphs, built with <u>very specific rules.</u>

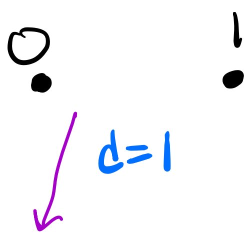
Def: Given $d \in \mathbb{N}$, the

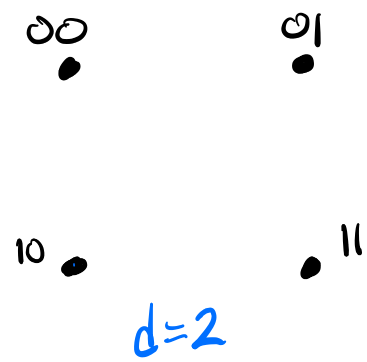
or " " is the graph with $G=(V,E)$ and

- $V = \{ \qquad \}$
- $E = \{ (u,v) \mid u,v \text{ differ in } \qquad \}$

Ex:



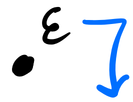




Notice: Each hypercube is like <u>two</u> copies of previous connected with some edges!

Recursive Def:

- 0-dimensional hypercube:



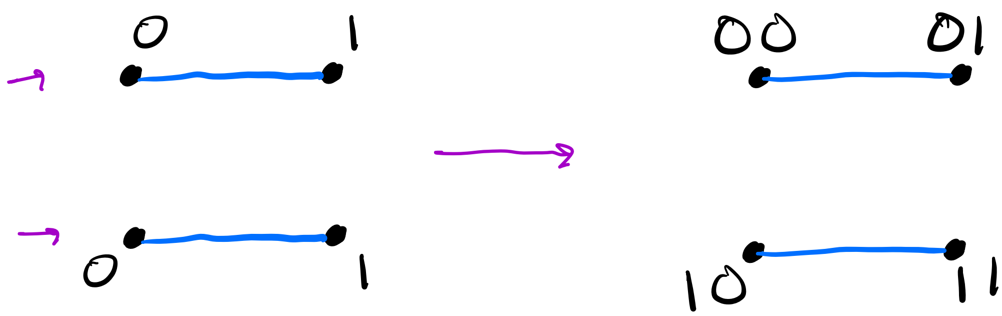
empty string

- 1-dimensional hypercube:

Take two 0-cubes, put 0 in front of vertices from one copy, 1 in front of other

Connect vertices that are copies of each other

- 2-dimensional hypercube:



Take two 0-cubes, put 0 in front of vertices from one copy, 1 in front of other

Connect vertices that are copies of each other

- d-dimensional hypercube: Take two -cubes, put in front of vertices of one copy, in front of other. Then, connect vertices that are copies.

Fact: These two definitions are the same.

Prop: The d-cube has vertices.

Direct Proof: $V = \{$ length d binary strings $\}$
(Bit string def)
and there are length d binary strings. $\square$

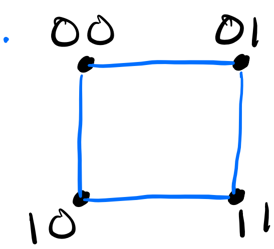
Inductive Proof: Base ($d=0$): Has vertices.
(Recursive def)

IH: Say d-cube has vertices.

IS: $(d+1)$-cube is two with extra edges, so has vertices. $\square$

Lemma: Every vertex in the d-cube has degree .

Proof: Each vertex is a string.



We put an edge when vertices differ by .

For each vertex, there are bit positions we could change, so we get touching. $\square$

Prop: The d-cube has edges.

Direct Proof: By Handshaking lemma,

$2|E| = \quad = \quad = \quad \Rightarrow |E| =$ $\square$

lemma

Induction Proof: Base $(d=0)$: $|E| =$.

IH: Say d-cube has edges.

IS: Consider $(d+1)$-cube. To make it, we take two d-cubes and connect duplicate vertices. So, we have (# edges in $(d+1)$-cube)

$= 2 \cdot (\qquad) + (\qquad)$

$= 2 \cdot (\qquad) + (\qquad) = \qquad = \qquad$ $\square$

Moral: For hypercube proofs, use

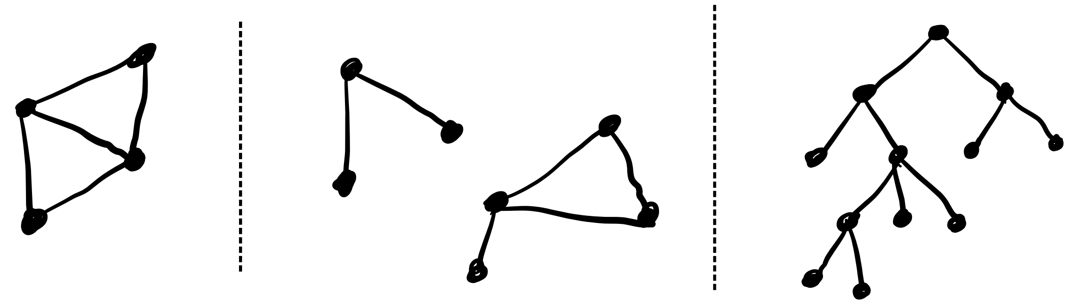
- Direct proof + bit string def

or

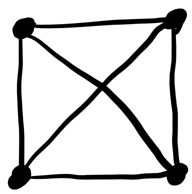
- Induction on dimension + recursive def

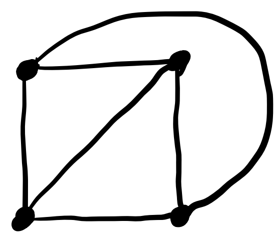
Planar Graphs

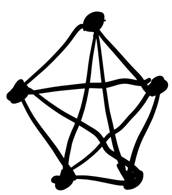
Def: G is if it can be drawn on a piece of paper .

Ex: 

Fact: Any tree is planar.

Q: Is  planar?

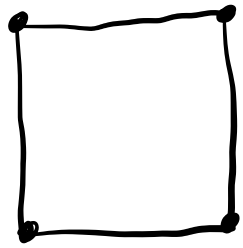
We can draw it as 

Q: Is  K_5 planar?

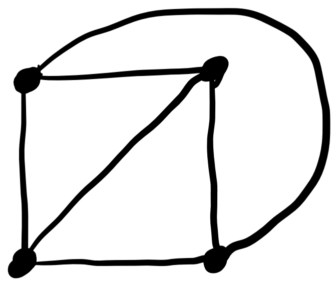
Harder to say... Let's come back later...

Let's study these closer!

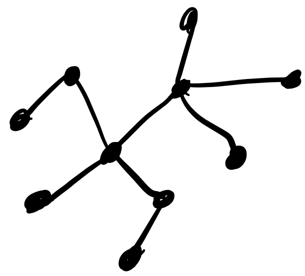
Def: The regions of the plane that a planar graph divides out are called .

Ex: 

\# vertices $= v =$
\# edges $= e =$
\# faces $= f =$

Ex: 

\# vertices $= v =$
\# edges $= e =$
\# faces $= f =$

Ex: 

\# vertices $= v =$
\# edges $= e =$
\# faces $= f =$

We get a pattern!

Theorem (Euler): If G is connected and planar,

.

Proof: By induction on e.

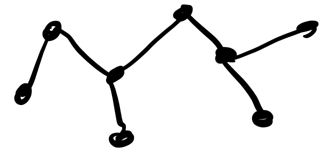
Base ($e=0$): $\bullet$ is only connected graph with . Then, and , so

.

IH: Suppose the statements holds when $e=k$ for some $k \geq 0$.

IS: Say $e=k+1$. We do two cases.

1. Say G is a tree.

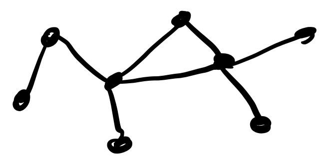


A tree has no cycles, so .

Also, $v =$ $=$.

So, $v - e + f =$ $=$.

2. If G is not a tree but is connected, it has a cycle.



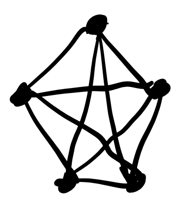
Delete an edge in the cycle.

This reduces by one, but also by one, since we combined the face with another face.

Let $e' = e - 1 = k$, $f' = f - 1$. By IH, $v - e' + f' =$. So, .

So, . □

Cool! Can we use this to tell if K_5 is planar?



No! To use $v - e + f = 2$, we must be able to count faces!

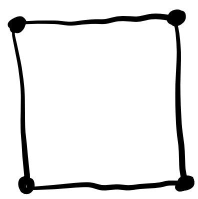
To count faces, we must already have a planar drawing.

Idea: Planar graphs must have "few" edges per vertex. Otherwise, get crosses.

Let's use $v-e+f=2$ to find inequality for and .

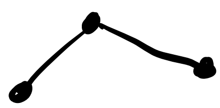
Step 1: Relate edges and faces.

Notice each face has a number of "sides."

Ex: 

f_1 has sides

f_2 has sides

Ex: 

f_1 has sides

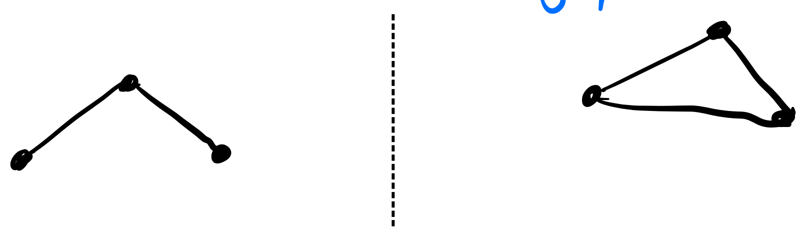
Notice: Each edge is a side for faces, or for face .

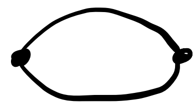
Handshaking Lemma for Faces: $\sum_{i=1}^{f} s_i = 2e$

Proof: Each edge acts as a side twice. $\square$

Step 2: Number of sides per face.

Consider a connected graph with ≥ 1 edge.



Recall we don't allow: 

So, we notice each face has ________!

Handshaking for faces: $2e = \sum_{i=1}^{f} s_i \geq \sum_{i=1}^{f} 3 = 3f$

Key Fact: If $e \geq 1$, $\boxed{\quad\quad\quad\quad}$.

Step 3: Use Euler.

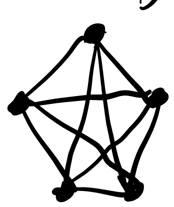
$v - e + f = 2 \overset{\times 3}{\Longrightarrow}$

$\overset{3f \leq 2e}{\Longrightarrow}$

$\Longrightarrow \boxed{\quad\quad\quad\quad\quad\quad}$

Prop: If G is connected and planar and $e \geq 1$,
then $e \leq 3v - 6$.

Ex: K_5

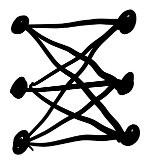


$v =$

$e =$

$e \overset{?}{\leq} 3v - 6$

Ex: $K_{3,3}$



$v=$

$e=$

$e \overset{?}{\leq} 3v-6$

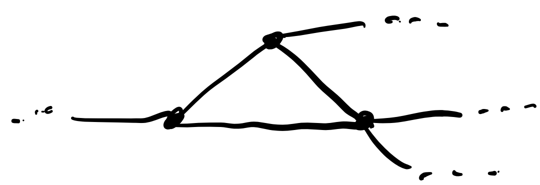
But... We never said $e \leq 3v-6 \Rightarrow$ planar!

Need more tests...

Lemma: If G is ______, connected and planar with $e \geq 1$ then each face has ≥ 4 sides.

Proof: Recall bipartite means we can split vertices in 2 groups with only edges <u>between</u> groups (never <u>within</u>).

Suppose G has a face with sides.



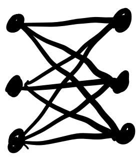
This part of G fails the bipartite condition! Contradiction! $\square$

- When we showed every face had $\geq$ sides we showed $\boxed{\qquad\qquad}$.
- If G is bipartite, every face has $\geq$ sides, so by same logic $\boxed{\qquad\qquad}$.

Plug into $v-e+f=2$. Get $e \le$.

Prop: If G is bipartite, connected and planar and $e>1$, then $e \le 2v-4$.

Ex: $K_{3,3}$



$v=6$

$e=9$

$e \overset{?}{\le} 2v-4$

Summary: K_5, $K_{3,3}$ not planar.

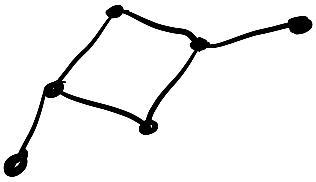
It turns out <u>all</u> non-planar graphs have K_5 or $K_{3,3}$ inside them in a "sneaky" way...

Out of scope!

Vertex Coloring

Idea: Generalize bipartite to >2 groups.

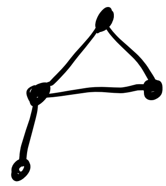
Def: A graph is if its vertices can be colored with so that all edges go between .

Ex: 

!

Also automatically ,

, etc...

Ex: 

, not .

Note: 2-colorable is the same as bipartite

The are just the .

Theorem: Every planar graph is 5-colorable.

Proof: Induction on v.

Base case $(v=0)$: Clear.

IH: Holds for k vertices, $k \geq 0$.

IS: Start with $k+1$ vertices. We want to remove a vertex with few edges.

Lemma: A planar graph has a vertex with degree ≤ 5.

Proof: If every vertex had degree $\geq$,

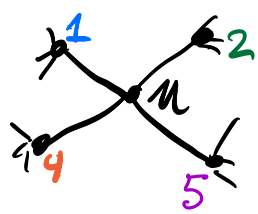
by Handshaking lemma, $2e = \sum_{v \in V} \deg v \geq$

So $e \geq$. But we know $e \leq 3v-6$!

Contradiction! $\square$

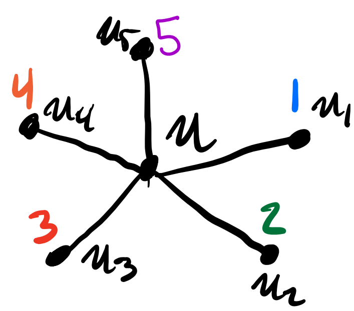
So, pick a vertex u of degree ≤ 5 and remove it. By IH, we can color the rest of the graph with 5 colors. Add u back. We must color u.

Case 1: If $\deg u < 5$, there is a color u is not touching, so use that. Done!



Case 2a: If $\deg u = 5$ but the 5 neighbors did not use all 5 colors, done like Case 1.

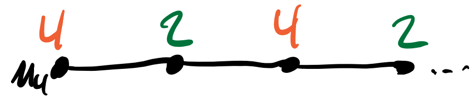
Case 2b: $\deg u = 5$, all 5 neighbors used different colors.

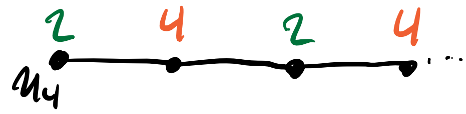


Try changing u_4 to color 2.

If this doesn't cause conflicts, done!

If something touching u_4 is already 2, make it 4, and so on.

Just switch chain of 

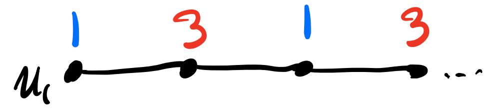
to 

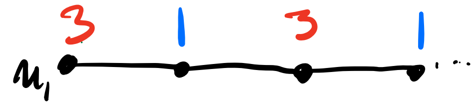
Done! Unless... the chain reaches u_2.

Then, we must switch u_2 to color 4

and we get same problem.

If this happens, switch u_1 to color 3

and do new chain 

to 

Again, done! Unless... the chain reaches u_3.

☆ But: We picked colors to swap so that if <u>both</u> bad things happen, the – path and – path cross!

The graph is planar, so the cross is at a vertex v. What color is v?

In 2 – 4 path $\Rightarrow$ must be 2 or 4.

In 1 – 3 path $\Rightarrow$ must be 1 or 3.

Contradiction! So, must be in some other case that already worked. □

We can do better.

Theorem: Every planar graph is 4-colorable.

Proof: In 1976 by Appel and Haken.

By cases: 1834 cases.

Done on a computer, took over 1000 hours.

Has since been simplified. $\square$

You may use this theorem without proof.