Today, we count things.

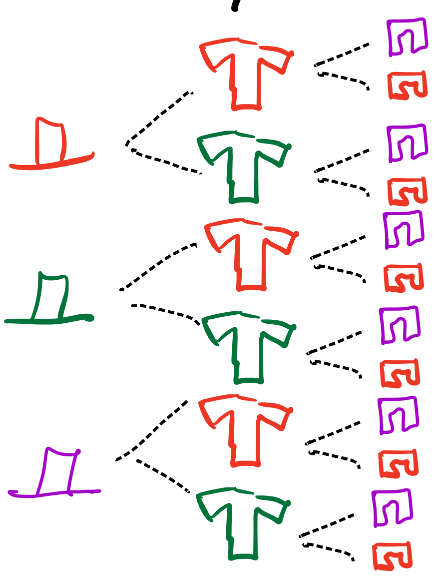
No real proofs, mainly examples

Ordered Counting

Ex: Picking an outfit

Bob has 3 hats, 2 coats, 2 pants.

How many outfits?



Choices: $3 \times 2 \times 2 = \boxed{12}$ outfits

First Rule of Counting: When making a sequence of k choices, if there are n_1 options for choice 1, n_2 options for choice 2 after making choice 1, and so on, there are $n_1 \cdot n_2 \cdots n_k$ total outcomes.

"Multiply consecutive choices"

Ex: Flip a coin, roll a 6 sided die, then flip 2 more coins. How many sequences of outcomes?

H/T 1-6 H/T H/T

Outcomes: $2 \times 6 \times 2 \times 2 = \boxed{48}$

Important case: All choices from same set.

Ex: Length k binary strings

0/1 0/1 0/1 0/1 .... 0/1

Choices: $2 \times 2 \times 2 \times 2 \times \cdots \times 2 = \boxed{2^k}$ strings

Ex: Length k word from size n alphabet

_ _ _ ... _

Choices: $n \times n \times n \times \cdots \times n = \boxed{n^k}$ words

General Rule: Given n elements and want an ordered sequence of k elements, with repeats allowed.

There are n^k ways to do this.

Two natural questions:

1. What if no repeats?
2. What if we don't care about order?

Make a table: _ _ _ . . . _

\# ways to pick k elements from set of n

Order Matters?	Repeat Elements (Replacement)? Yes	No
Yes	n^k	?
No	?	?

Order Matters, No Replacement

Ex: Anagram the word mod

_ _ _

- m
  - o — d
  - d — o
- o
  - m — d
  - d — m
- d
  - m — o
  - o — m

Choices: $3 \times 2 \times 1 = 3! = \boxed{6}$ anagrams

Important Ex: Number of ways to order n distinct objects.

Choices: $\overline{n} \times \overline{(n-1)} \times \overline{(n-2)} \times \cdots \times \overline{1} = \boxed{n!}$ orders

What if there are more objects than spaces?

Ex: Number of ways to pick a sequence of 5 different cards from a standard 52 card deck.

Choices: $\overline{52} \times \overline{51} \times \overline{50} \times \overline{49} \times \overline{48} = \boxed{\dfrac{52!}{47!}}$ sequences

convenient notation

General Rule: Given n elements and want an ordered sequence of k elements, with no repeats allowed.

There are $n(n-1)\cdots(n-k+1) = \boxed{\dfrac{n!}{(n-k)!}}$ ways to do this.

\# ways to pick k elements from set of n

Repeat Elements (Replacement)?

Order Matters?	Yes	No
Yes	n^k	$\dfrac{n!}{(n-k)!}$
No	?	?

Unordered Counting

Ex: Anagrams of eggs

4 letters, so 4! No...

Problem: This counting assumes we can distinguish between the g's.

Not so easy to count like before...

e/g/s __ __ __ __

3 3 or 2 ?? → If we used g first, still have 3 choices

Otherwise, only have 2...

Trick: First assume the g's are different.

Get 4! orders

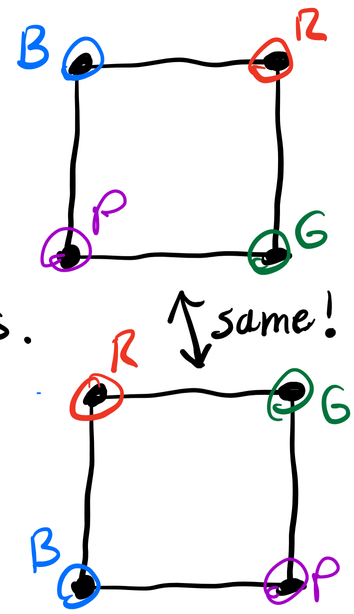
Then, notice we have exactly twice the real number of anagrams... for each anagram, we counted it again with g's switched. So, divide by 2.

$$\Rightarrow \frac{4!}{2} = \boxed{12} \text{ anagrams}$$

General Trick ("Second Rule of Counting"):

When counting, we can overcount by a constant factor, then divide by that factor.

Ex: Coloring a square.



We want to color each vertex a different color out of 4 colors.

But, we'll say two colorings are the same if they are rotations of each other.

Suppose all colorings are different, count colorings.

Choices: $\overset{\text{top left}}{\underline{4}} \times \overset{\text{top right}}{\underline{3}} \times \overset{\text{bottom left}}{\underline{2}} \times \overset{\text{bottom right}}{\underline{1}} = 4!$

Now, notice we counted each actually different coloring 4 times, since the square has 4 different orientations. So, divide by 4.

$$\Rightarrow \frac{4!}{4} = \boxed{6} \text{ colorings}$$

Ex: Number of ways to draw a hand of 5 cards from a 52 card deck.

Note: Now, we don't care about the order.

Suppose we do care about order

$\overline{52}\cdot\overline{51}\cdot\overline{50}\cdot\overline{49}\cdot\overline{48} = \frac{52!}{47!}$ hands.

We have overcounted by a factor of $5!$ since this is the number of ways to order a 5 card hand. So, divide by $5!$.

$$\Rightarrow \boxed{\frac{52!}{5!\cdot 47!}} \text{ hands}$$

General Rule: Given n elements and want an unordered collection of k elements, with no repeats allowed.

There are $\frac{n!}{k!(n-k)!}$ → number of sequences with order

↳ number of orders per unordered collection

ways to do this.

Def: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ "n choose k"

As we just saw, there are $\binom{n}{k}$ ways to choose k distinct items from a set of n without order.

ways to pick k elements from set of n

Repeat Elements (Replacement)?

Order Matters?	Yes	No
Yes	n^k	$\frac{n!}{(n-k)!}$
No	?	$\binom{n}{k}$

Ex: Have 7 different candies, want to pick 5 to eat now.

$\Rightarrow \binom{7}{5}$ possibilities

Want to pick 2 to <u>not</u> eat now.

$\Rightarrow \binom{7}{2}$ possibilities

Should be the same... picking 5 to eat now is same as picking 2 to not eat now!

Indeed, $\binom{7}{5} = \frac{7!}{5!(7-5!)} = \underbrace{\frac{7!}{5!\,2!}}_{=21} = \frac{7!}{2!(7-2)!} = \binom{7}{2}$

In general: $\binom{n}{k} = \binom{n}{n-k}$

"# ways to take k = # ways to leave $n-k$"

Next: Some sneakiness.

Ex: How many ordered sequences of 10 coin flips have exactly 3 heads?

H/T H/T H/T _ _ _ _ _ _ _

Choices: $2 \times 2 \times 2 \times$? ?

At some point, we run out of heads or tails, but where? Can't count like this!

This is unordered counting in disguise...

Notice: We can get each sequence by choosing 3 of 10 slots to be heads. Then, put tails in other 7 slots.

$\Rightarrow \binom{10}{3}$ sequences

We could have chosen the 7 slots for tails instead. Luckily, $\binom{10}{7} = \binom{10}{3}$.

Same thing!

Moral: Counting techniques can appear in unlikely places! Even though we were counting ordered sequences, the special condition made the answer be in terms of $\binom{n}{k}$, an object from unordered counting.

Stars and Bars

Ex: You want to get a box of 6 donuts from a shop with 3 types.

Can get many of same type, order doesn't matter.

Attempt 1: Count with order, divide.

$$\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$$
$$3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 \text{ with order.}$$

Problem: What to divide by? Depends on how many donuts are the same...

If all donuts are the same, no overcounting!

If many are different, we did overcount!

This doesn't work...

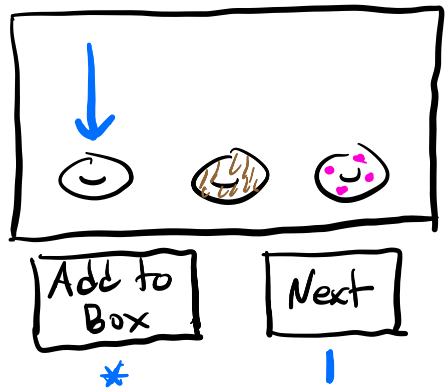
Idea: Count a different set of the same size.

Zeroth Rule of Counting: If there is a bijection $A \to B$, then $|A| = |B|$.

Goal: Find a bijection from the set of donut boxes to some other set.

To do this, we will make a system for filling our box.

The Donut Machine: An Ordering Device

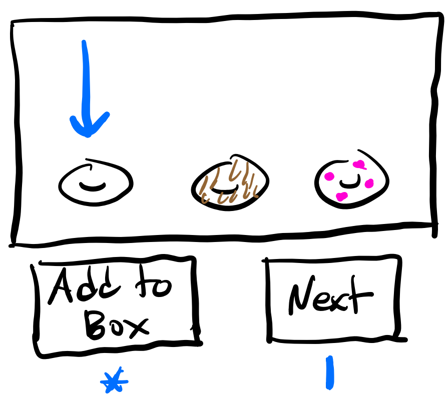


Rules to make a box

1. Pointer starts at the leftmost donut type.
2. Press Add to Box to add the currently selected donut to your box. Press Next to go to the next type.
3. You can add as many of each type as you want (until your box is full), but once you hit next, there is no going back.
4. You must hit buttons until your box is full and the pointer is at the rightmost donut type.

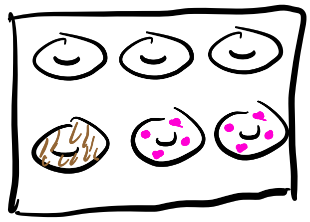
Point 1: To fill a box of 6 donuts with 3 types, we must press Add to Box 6 times and Next 2 times, for a total of 8 presses.

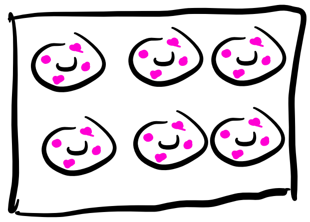
Ex: Translating between boxes and button sequences

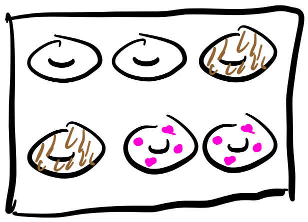


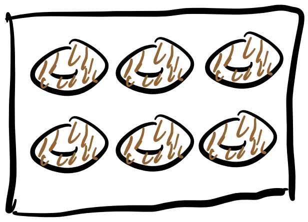
Use * to denote Add to Box.

Use | to denote Next.

* * * | * | * * → 

| | * * * * * * → 

 → * * | * * | * *

 → | * * * * * * |

Point 2: Every sequence of 6 *'s and 2 |'s gives a different box of 6 donuts with a mix of 3 types, and we can get all boxes this way.

This function boxes ⟷ button sequences is a bijection.

So, # boxes = # button sequences.

How many button sequences? _ _ _ _ _ _ _ _

Have length 8 sequences with 6 *, 2 |.

Same as coin flips! Just choose which locations have * $\Rightarrow \boxed{\binom{8}{6}}$.

General Rule: Given n elements and want an unordered collection of k elements, with repeats allowed.

(n types of donuts, want box of k).

Need to press Add to Box k times (k $*$'s).

Need to press Next $n-1$ times ($n-1$ $|$'s).

So, count # sequences of length $k+n-1$ with k $*$'s and $n-1$ $|$'s.

Just pick where $*$'s go!

$$\Rightarrow \boxed{\binom{k+n-1}{k}}$$

ways to pick k elements from set of n

Repeat Elements (Replacement)?

Order Matters?	Yes	No
Yes	n^k	$\dfrac{n!}{(n-k)!}$
No	$\binom{k+n-1}{k}$	$\binom{n}{k}$

Warning: This table is a great start, but it does <u>not</u> cover all types of counting problems you will see. You may still need to get creative!

Next Time

- Combinatorial Proofs
- Inclusion-Exclusion
- More examples