Review

\# ways to pick k elements from set of n

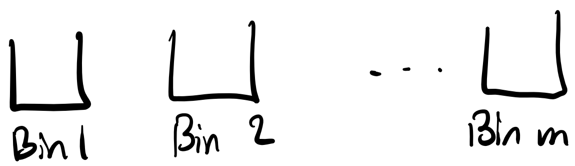
	Repeat Elements (Replacement)? Yes	No
Order Matters? Yes	n^k	$\frac{n!}{(n-k)!}$
No	$\binom{k+n-1}{k}$	$\binom{n}{k}$

Ex: Counting with balls and bins

Suppose we have n balls, m bins

We throw each ball into a bin.

How many outcomes?



1. Assume balls are numbered.

Ball label:	1	2	3	4	....	n	
Choices:	×	×	×	×		×	$= \boxed{}$

2. Assume balls are identical.

Now, just care about how many balls each bin has.

View as stars and bars problem:

Start with pointer at Bin 1

$*$ = add a ball (balls = donut box spaces)

$|$ = go to next bin (bins = donut types)

$\Rightarrow$ $*$'s, $|$'s

$\Rightarrow$ $\boxed{}$ outcomes

The Binomial Theorem

Recall: $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ "n choose k"

This type of expression is also called a . Let's see why.

Def: An expression $(a+b)^n$ with $n \in \mathbb{N}$ and is called a binomial.

Ex: $(a+b)^2 =$

$(a+b)^3 =$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

This pattern continues!

Binomial Theorem: For all $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Proof: Expand the lefthand side as

$$(a+b)(a+b)(a+b)\cdots(a+b).$$

When multiplied out, each term will look like for some k since we have terms and can take an or from each.

How many times does each term appear?

\# ways to get $a^k b^{n-k}$ = \# ways to pick from terms, from the rest

So, ways to get $a^k b^{n-k}$. Summing all terms, we get . $\square$

Corollary: For all $n \in \mathbb{N}$, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Proof: We showed $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

Plug in $a=$, $b=$. $\square$

Inclusion-Exclusion

We learned to multiply for consecutive choices.

How about either/or type choices?

Ex: Restaurant has 5 starters, 4 entreés, and 2 desserts. You want an entreé and either a starter or dessert.

How many meals?

$= \boxed{}$ meals

General Rule: When a counting problem has several cases which don't overlap, to get the total number.

What about overlap?

Ex: How many numbers are divisible by 2 or 3 in the range $\{1, \dots, 24\}$?

Divisible by 2 :

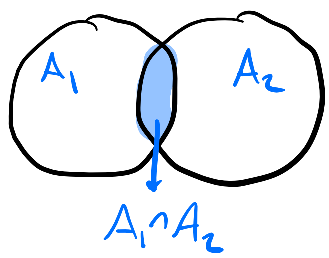
Divisible by 3 :

Divisible by both :

these are double counted!

$\Rightarrow \qquad = \square$

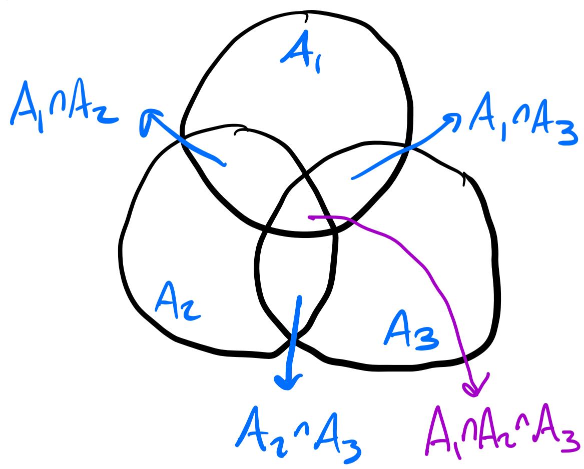
Simple Inclusion Exclusion: If A_1 and A_2 are finite sets, then



$|A_1 \cup A_2| =$

remove double counted

How about 3 sets?



$|A_1 \cup A_2 \cup A_3| =$

Can we generalize?

Theorem: Let $A_1, \ldots, A_n \subseteq A$. Then,

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = k}} \left| \bigcap_{i \in S} A_i \right|.$$

So, we add all

subtract all with $i<j$

add all with $i<j<k$

$\vdots$

Proof: Consider any $a \in A$. Our goal is to to show that if $a \in A_1 \cup \cdots \cup A_n$, then a is counted exactly once on the right and if $a \notin A_1 \cup \cdots \cup A_n$ then a is not counted on the right.

First, suppose $a \notin A_1 \cup \cdots \cup A_n$. Then a is not in any of the sets on the right side, so it is not counted on the right.

Next, suppose $a \in A_1 \cup \cdots \cup A_n$.

Let $M = \{ i \mid a \in A_i \}$. Know $m = |M| \geq$.

Then, notice $a \in \bigcap_{i \in S} A_i \iff S \subseteq M$.

So, to find how many times a is

counted on the right, we put a 1 for each set $\bigcap_{i \in S} A_i$ with $S \subseteq M$.

$$\Rightarrow \# \text{ times } a \text{ counted } = \sum_{k=1}^{m} (-1)^{k-1} \sum_{\substack{S \subseteq M \\ |S| = k}} 1$$

$$= \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k}$$

We showed $\sum_{k=0}^{m} (-1)^k \binom{m}{k} = 0$.

$$\Rightarrow 1 + \sum_{k=1}^{m} (-1)^k \binom{m}{k} = 0.$$

$$\Rightarrow 1 = -\sum_{k=1}^{m} (-1)^k \binom{m}{k} = \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k}$$

So a is counted once! $\square$

Let's do an application.

Derangements

Recall: $n!$ ways to order n distinct objects.

"Permutations"

Def: A derangement is a permutation where no element ends in the position it started in.

Ex: Anagrams of wolf

fwol is a derangement

fowl is not: o is a "fixed point"

How can we count derangements of n objects?

Let A_i be the permutations where object i is a fixed point.

Then, $A_1 \cup A_2 \cup \cdots \cup A_n$ are the permutations which are not derangements.

$\Rightarrow$ # derangements $= n! - |A_1 \cup A_2 \cup \cdots \cup A_n|$

So, just count $|A_1 \cup A_2 \cup \cdots \cup A_n|$!

We know $|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = k}} \left| \bigcap_{i \in S} A_i \right|$.

How big is $\bigcap_{i \in S} A_i$ when $|S| = k$?

$\bigcap_{i \in S} A_i$ are the permutations that have fixed points at object i for each $i \in S$.

$|S|=$, so we just count the number of ways to order the other objects.

$$\Rightarrow \left|\bigcap_{i\in S} A_i\right| =$$

$$\Rightarrow |A_1\cup\cdots\cup A_n| = \sum_{k=1}^{n}(-1)^{k-1}\sum_{\substack{S\subseteq\{1,\dots,n\}\\|S|=k}}\left|\bigcap_{i\in S}A_i\right|$$

$$= \sum_{k=1}^{n}(-1)^{k-1}\sum_{\substack{S\subseteq\{1,\dots,n\}\\|S|=k}}(n-k)!$$

subsets $S\subseteq\{1,\dots,n\}$ of size k

$$= \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)!$$

$$= \sum_{k=1}^{n}(-1)^{k-1}\frac{n!}{k!(n-k)!}(n-k)!$$

$$= n!\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}$$

$$\Rightarrow \#\text{ derangements} = n! - |A_1\cup A_2\cup\cdots\cup A_n|$$

$$= n! - n!\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}$$

$$= n!\frac{(-1)^0}{0!} + n!\sum_{k=1}^{n}\frac{(-1)^k}{k!} = \boxed{n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}}$$

Combinatorial Proofs

Idea: Show two expressions are the same by counting.

Ex: Show $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

Come up with a counting word problem.

Show and are both solutions to the word problem.

Tip: Start with simpler side to come up with the problem.

Problem: A company has n applicants, and can accept/reject any number of them. How many ways to decide which to accept?

LHS: There are 2 possibilities for each of n applicants, so 2^n combinations.

RHS: We do cases based on how many people the company wants.

0 hires: combinations

1 hire: combinations

⋮

n hires: combinations

In total, Combinations.

So, $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ $\square$

General Rules: To convert a formula to a story:

$\times \longrightarrow$

$+ \longrightarrow$

$\binom{n}{k} \longrightarrow$

$n! \longrightarrow$

Ex: Show $\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k}{k}$

Problem: Pick $k+1$ numbers from $\{1, 2, \ldots, n\}$

LHS: Number of ways to pick $k+1$ from n is $\binom{n}{k+1}$.

RHS: Cases based on the smallest number we take

Smallest is 1: ways to pick k more numbers from $\{ \quad \}$

Smallest is 2: ways to pick k more numbers from $\{ \quad \}$

$\vdots$

Smallest is $n-k$: ways to pick k more numbers from $\{ \quad \}$

Smallest can't be bigger than $n-k$ if we want to pick k things.

In total, ways to pick $k+1$ numbers from $\{1, \ldots, n\}$.

So, $\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k}{k}$. $\square$

Summary

Thanks for learning discrete math with me!

If you enjoyed....	You should try ...
Sets/Functions/Countability	Math 104 (Real Analysis)
Computability	CS 172 (Computability)
Graphs	CS 170 (Algorithms)
Modular Arithmetic + Polynomials	Math 113 (Abstract Algebra)
	Math 115 (Number Theory)

RSA	CS 161 (Security) CS 171 (Cryptography) Math 116
Counting	Math 172 (Combinatorics) Rest of CS 70!

Enjoy probability!