Lecture 4C: Conditional Probability (II)

UC Berkeley CS70
Summer 2023
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Combinations of Events

$A \cap B$

As we've learned, we can look at **intersections of events** together, and find the probability.

On Monday, I said for a coin where P(heads) = ⅓:

P(HHHT) = $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}$ or $\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)$

Why are we allowed to do this?

Magic coin: every time I flip a heads, P(heads) ↑

P(HHHT)

Independence

Two events are **independent** if $P(A \mid B) = P(A)$

A given B

- That is, because A doesn't depend on B, the probability that A happens **given B happened** is the same as the probability that A happens.

independent: don't affect each other

Starting lecture at 12:30

→ Zamboni at Cupertino Ice Center

clears ice at 11:30

Expand Independence via Bayes' Rule

$\boxed{P(A \mid B) = P(A)}$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

← Bayes' Rule

$$\underline{P(B \mid A)} = \frac{P(A \cap B)}{P(A)}$$

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

$$\boxed{P(A \cap B) = P(A)P(B)}$$

→ independence equation

$$\boxed{\frac{P(A \cap B)}{P(A)} = P(B)}$$

$$\boxed{P(B \mid A) = P(B)}$$

Some Independence Practice

0.3 of being tails

P(heads) = 0.6

You have two weighted coins. Coin 1 has probability **0.6** of coming up heads, and Coin 2 has probability **0.7** of coming up heads. The two coins are independent.

P(heads) = 0.7

$$\boxed{P(A \cap B) = P(A)P(B)}$$

A: event Coin 1 is a H

B: event Coin 2 is a H

What's the probability of Coin 1 & Coin 2 being heads?

$$P(\text{Coin } 1 = H \cap \text{Coin } 2 = H) = P(\text{Coin } 1\ H)\,P(\text{Coin } 2\ H)$$

$$= 0.6 \times 0.7$$

Coin 1 Coin 2

	H	T
H	0.6×0.7	0.6×0.3
T	0.4×0.7	0.4×0.3

HT

Some Independence Practice

Are the events "your dice roll is even" and "your dice roll is odd" **independent events**?

$A: \{2, 4, 6\}$

$B: \{1, 3, 5\}$

$P(A) = \frac{3}{6} = \frac{1}{2}$

$P(B) = 3/6 = 1/2$

$P(A \cap B) = \emptyset = 0$

$0 \neq 1/4$

$0 = 1/2 \times 1/2$

not independent

Some Independence Practice

Are the events "your dice roll is 1" and "your dice roll is odd" **independent events**?

$A: \{1\}$

$P(A) = \frac{1}{6}$

$B: \{1, 3, 5\}$

$P(B) = \frac{3}{6} = \frac{1}{2}$

$P(A \cap B) = \{1\} = \frac{1}{6}$

$\frac{1}{6} \neq \frac{1}{6} \cdot \frac{1}{2}$

dependent events

Some Independence Practice

Are the events "your dice roll is a 1 or 2", and "your dice roll is odd" **independent events**?

$A: \{1, 2\}$

$P(A) = \frac{2}{6} = \frac{1}{3}$

$B: \{1, 3, 5\}$

$P(B) = \frac{3}{6} = \frac{1}{2}$

$A \cap B: \{1\}$, $P(A \cap B) = \frac{1}{6}$

$\frac{1}{6} = \frac{1}{3} \cdot \frac{1}{2}$, $\frac{1}{6} = \frac{1}{6}$

independent

Mutual Independence

$P(A \cap B) = P(A)P(B)$

We can expand beyond just two events. $P(A \cap B \cap \ldots A_n) = P(A)P(B)P(C) \ldots P(A_n)$

Definition 14.4 (Mutual independence). *Events $A_1, \ldots, A_n$ are said to be* mutually independent *if for* **every** *subset $I \subseteq \{1, \ldots, n\}$ with size $|I| \geq 2$,*

$$\mathbb{P}[\bigcap_{i \in I} A_i] = \prod_{i \in I} \mathbb{P}[A_i]. \tag{6}$$

Back to coin tosses! How do we find P(HHHT) using the above definition?

$P(HHHT) = P(H \cap H \cap H \cap T)$

$= P(H)P(H)P(H)P(T) = (1/3)^3(2/3)$

Pairwise Independent but **not** Mutually Independent?

set of two events, they're independent

assume coin flips independent

all of the events together are independent

We flip a fair coin twice. Let A be the event that the first flip is H, B be the event that the second flip is H, and C be the event that both flips are the same.

$P(A \wedge B) = P(A)P(B)$ ✓

$P(B \wedge C) = P($first flip is heads & my second flip is a heads$) = \left(\frac{1}{2}\right)^2 = P(B)P(C)$

A, B, C

C is dependent on A & B in combination

$P(A \wedge B \wedge C) = P(A \wedge B)$

$P(A)P(B)P(C)$

Product Rule

What if they're not independent? We can use Bayes' Rule to get an equation for P(A ∩ B).

$$P(A \mid B) = \frac{P(A \wedge B)}{P(B)} \rightarrow \text{Bayes' Rule}$$

$$P(A \wedge B) = \boxed{P(A \mid B)P(B)}$$

→ not independence.

$$P(A \mid B) = P(A) \quad\rightarrow\quad = P(A)P(B)$$

Product Rule (cont.)

More generally, for any events $A_1, \ldots, A_n$,

$$\mathbb{P}[\bigcap_{i=1}^{n} A_i] = \mathbb{P}[A_1] \times \mathbb{P}[A_2 \mid A_1] \times \mathbb{P}[A_3 \mid A_1 \cap A_2] \times \cdots \times \mathbb{P}[A_n \mid \bigcap_{i=1}^{n-1} A_i].$$

Proof of the general product rule!

Product Rule (cont.)

Proof of the general product rule!

Probability of a Flush

$\binom{32}{5}$

$$P(\text{flush}) = 4 \cdot P\left(\begin{array}{c}\text{flush}\\ \text{in the}\\ \underline{\text{spades suit}}\end{array}\right)$$

Product Rule :)

$$P\left(\begin{array}{c}\text{five cards}\\ \text{being}\\ \text{a flush}\\ \text{in spades}\end{array}\right) = P\left(\begin{array}{c}\text{1st}\\ \text{spade}\end{array} \wedge \begin{array}{c}\text{2nd}\\ \text{spades}\end{array} \dots \wedge \begin{array}{c}\text{5th}\\ \text{spades}\end{array}\right)$$

$$P(\text{1st spade}) \times P\left(\begin{array}{c}\text{2nd}\\ \text{spade}\end{array} \middle| \begin{array}{c}\text{1st}\\ \text{spade}\end{array}\right) \times P\left(\begin{array}{c}\text{3rd}\\ \text{spade}\end{array} \middle| \text{1, 2 spades}\right)$$

$$\times P\left(\begin{array}{c}\text{4th}\\ \text{spade}\end{array} \middle| \text{1, 2, 3 spades}\right) \times P\left(\begin{array}{c}\text{5th}\\ \text{spade}\end{array} \middle| \begin{array}{c}\text{1, 2, 3}\\ \text{, 4 spades}\end{array}\right)$$

$$4 \times \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48} = \cancel{4} \frac{\binom{13}{5}}{\binom{52}{5}}$$

Monty Hall Revisited (with Product Rule)

Let C_i be the event that the contestant chooses door i, let P_i be the event that the prize is behind door i, and let H_i be the event that the host chooses door i.

$$P(P_1 \cap C_2 \cap H_3) = P(P_1)P(C_2 \mid P_1)P(H_3 \mid P_1, C_2)$$

$$\to P(P_1)P(C_2)P(H_3)$$

$$= \frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \frac{1}{9}$$

$$\neq \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$$

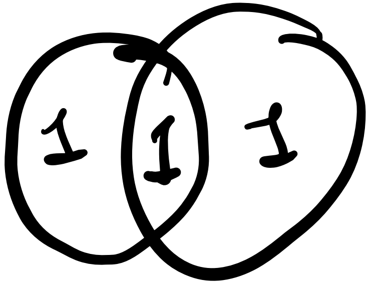
Monty Hall Revisited

Let C_i be the event that the contestant chooses door i, let P_i be the event that the prize is behind door i, and let H_i be the event that the host chooses door i.

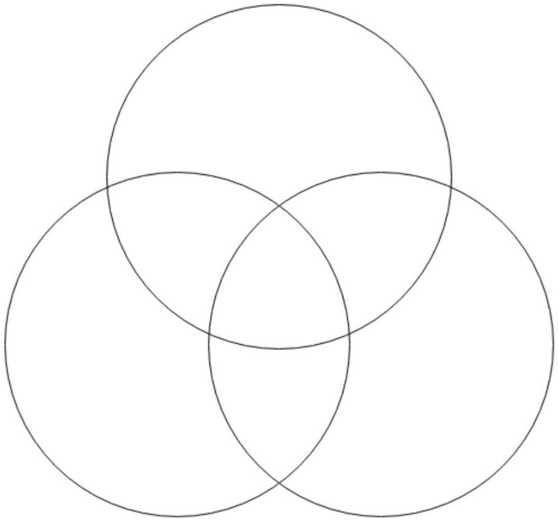
Unions of Events

Principle of Inclusion-Exclusion (but in probability!)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



What if you have three events?

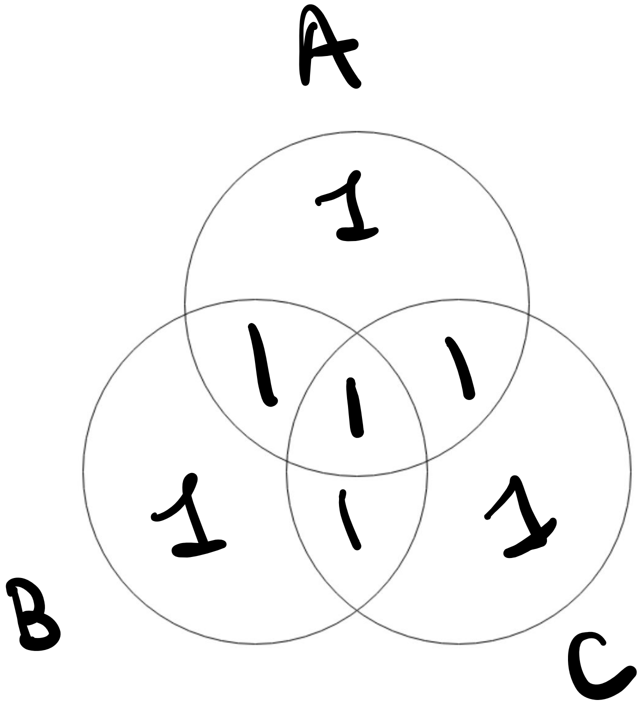


Unions of Events

Three events?

$$P(A \cup B \cup C)$$

$$P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(A \cap C)$$
$$- P(B \cap C) + P(A \cap B \cap C)$$



General Formula for Inclusion-Exclusion

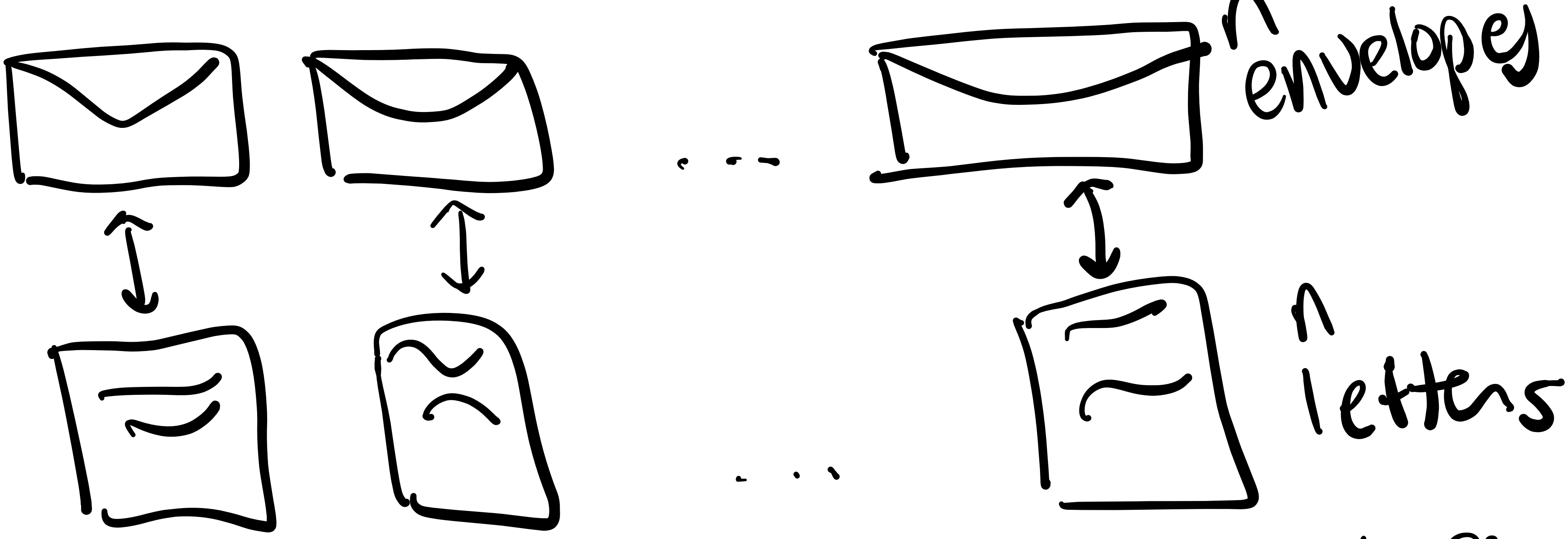
Theorem 14.2 (Inclusion-Exclusion). *Let $A_1, \ldots, A_n$ be events in some probability space, where $n \geq 2$. Then, we have*

$$\mathbb{P}[A_1 \cup \cdots \cup A_n] = \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq \{1,\ldots,n\} : |S| = k} \mathbb{P}[\textstyle\bigcap_{i \in S} A_i]. \tag{8}$$

$$P(A \cup B \cup C) = \sum P(A) - \sum P(A \cap B) + \sum P(A \cap B \cap C)$$

PIE Matching Problem

The matching problem

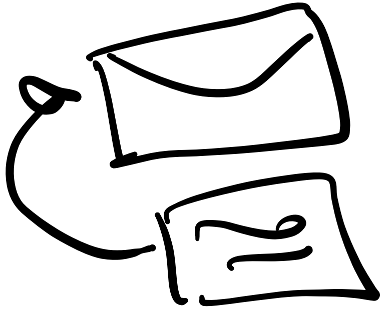


randomly permuting letters into envelopes

PIE Matching Problem

locations 1, ..., k

P(matches at k locations):



$n-1$ ... $n-1$] order however you want

P(envelope 1 got letter 1): $1 \cdot \frac{(n-1)!}{n!} = \frac{1}{n}$

P(envelope 1 gets letter 1 & envelope 2 gets letter 2): $1 \cdot 1 \cdot \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$

P(matches at location $1, \ldots, k$) $= \frac{(n-k)! \leftarrow |A|}{n! \leftarrow |\Omega|}$

Probability of No Matches

A_i: event that there is a match at position i

P(no match) = $1 - P(\text{at least one match})$

$P(\text{at least one match}) = P(\bigcup_{i=1}^{n} A_i)$

$$P(\bigcup_{i=1}^{n} A_i)$$

$P(A_i \cap A_j)$

$$= \sum_{i=1}^{n} P(A_i) - \sum\sum_{1\le i<j\le n} P(A_iA_j) + \sum\sum\sum_{1\le i<j<k\le n} P(A_iA_jA_j) - \cdots + (-1)^{n+1} P(A_1A_2\ldots A_n)$$

$= n \cdot \frac{1}{n} - \binom{n}{2} \cdot \frac{1}{n(n-1)} + \quad \ldots \quad (-1)^{n+1}\binom{n}{n}\frac{1}{n!}$

$1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} \quad \ldots \right)$

$= e^{-1}$ 36.8%

Union Bound

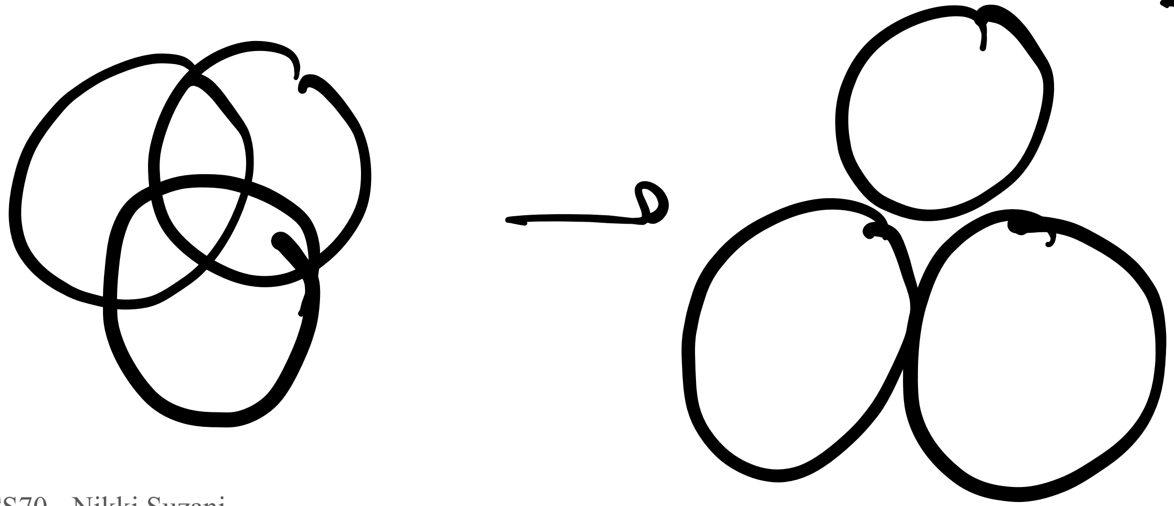
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$\leq P(A) + P(B)$

2. (**Union bound**) Let $A_1, \ldots, A_n$ be events in some probability space. Then, for all $n \in \mathbb{Z}^+$,

$$\mathbb{P}[\cup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} \mathbb{P}[A_i]. \tag{9}$$

$=$

This is also known as Boole's inequality! (Equal when **mutually exclusive**)



Proof of Union Bound

2. (**Union bound**) Let $A_1, \ldots, A_n$ be events in some probability space. Then, for all $n \in \mathbb{Z}^+$,

$$\mathbb{P}[\cup_{i=1}^{n} A_i] \le \sum_{i=1}^{n} \mathbb{P}[A_i]. \quad (9)$$

$B = \bigcup_{i=1}^{k} A_i$

1 $n=1$ $P(A_1) \le P(A_1)$ ✓

2 $n=k$ hypothesis $P\left(\bigcup_{i=1}^{k} A_i\right) \le \sum_{i=1}^{k} P(A_i)$

3 $n=k+1$ $P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(A_{k+1} \cup \bigcup_{i=1}^{k} A_i\right)$

$$= P(A_{k+1} \cup B) = P(A_{k+1}) + P(B) - \underline{P(A_{k+1} \cap B)}$$

$$\leq P(A_{k+1}) + P(B) \leq P(A_{k+1}) + \sum_{i=1}^{k} P(A_i)$$

$$= \sum_{i=1}^{k+1} P(A_i)$$

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(A_{k+1} \cup \underline{\bigcup_{i=1}^{k} A_i}\right)$$

$$B := \bigcup_{i=1}^{k} A_i \qquad P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

$$P(A_{k+1} \cup B) \leq P(A_{k+1}) + \underline{P(B)}$$

$$\leq P(A_{k+1}) + \sum_{i=1}^{k} P(A_i)$$

$$= \underline{\sum_{i=1}^{k+1} P(A_i)}$$

$= P(A_{k+1}$

More bounds

We can also **lower bound** the chance of a union:

$P(A_1)$ $P(A_2)$ $P(A_3)$

$P(A_1 \cup A_2 \cup A_3) = 0$

$= \max\{P(A_1), P(A_2), P(A_3)\}$

Bounding KPop Stans

A_1: event they like blackpink

A_2: event they like New Jeans

A_3: event they like twice

We know 30% of the class likes Blackpink, 20% of the class likes New Jeans, and 10% of the class likes Twice. Let's bound the probability that a randomly picked student likes at least one of the three KPop groups:

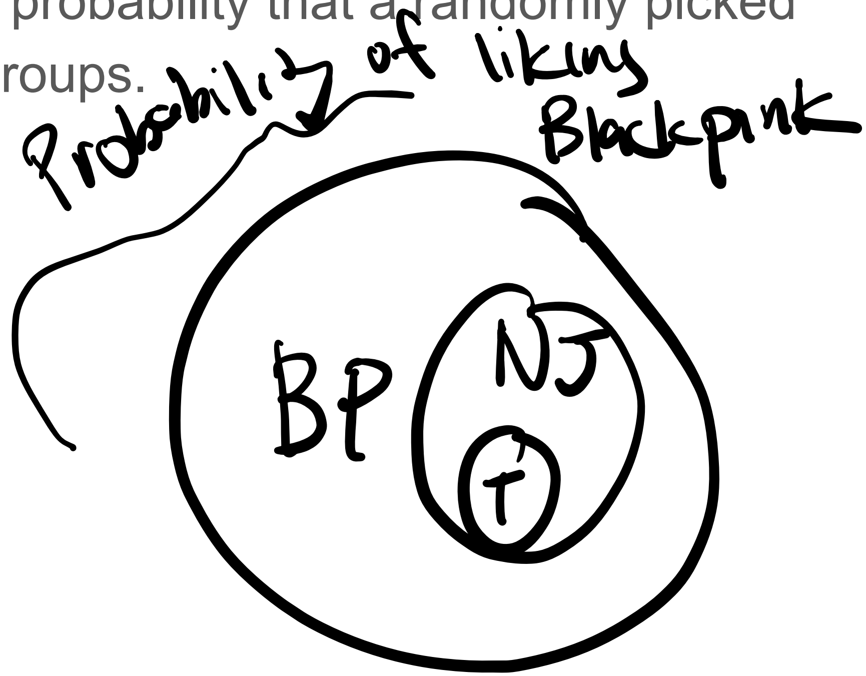
Lower bound: $\max\{P(A_1), P(A_2), P(A_3)\}$

Upper bound: Union bound

Bounding KPop Stans

We know 30% of the class likes Blackpink, 20% of the class likes New Jeans, and 10% of the class likes Twice. Let's bound the probability that a randomly picked student likes at least one of the three KPop groups.

Lower bound: 30%



Bounding KPop Stans

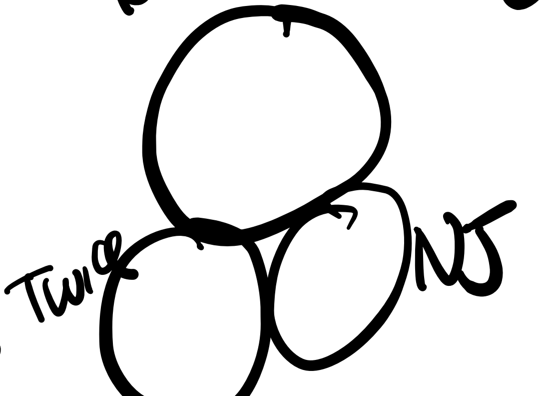
We know 30% of the class likes Blackpink, 20% of the class likes New Jeans, and 10% of the class likes Twice. Let's bound the probability that a randomly picked student likes at least one of the three KPop groups.

Upper Bound: $P(A_1 \cup A_2 \cup A_3) \le P(A_1) + P(A_2) + P(A_3)$

BP

$\le 60\%$

$= 60\%$ mutually exclusive case



Recap

pairwise vs. mutual

We learned about **independence** and **intersections of events**

- Concepts of mutual independence and the product rule

at lec 7

We also learned about **bounding** and **unions**

- The principle of inclusion-exclusion (in probability)
- How to upper and lower bound the value of a union

Notes

& their probabilities

For Lecture 4D, for Note 15 & 19 we're only going to be going over the definition of a random variable & different types of random variables. So feel free to ignore all the parts that talk about **expectation** until next week :)

$$P(A \cup B) \leq \underline{P(A) + P(B)}$$

$$P(A \cup B) = P(A) + P(B) \cancel{- P(A \cap B)}$$