Lecture 4D: Random Variables

UC Berkeley CS70
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What is a random variable?

Definition 15.1 (Random Variable). *A* <u>*random variable*</u> *X on a sample space Ω is a function $X : \Omega \to \mathbb{R}$ that assigns to each sample point $\omega \in \Omega$ a real number $X(\omega)$.*

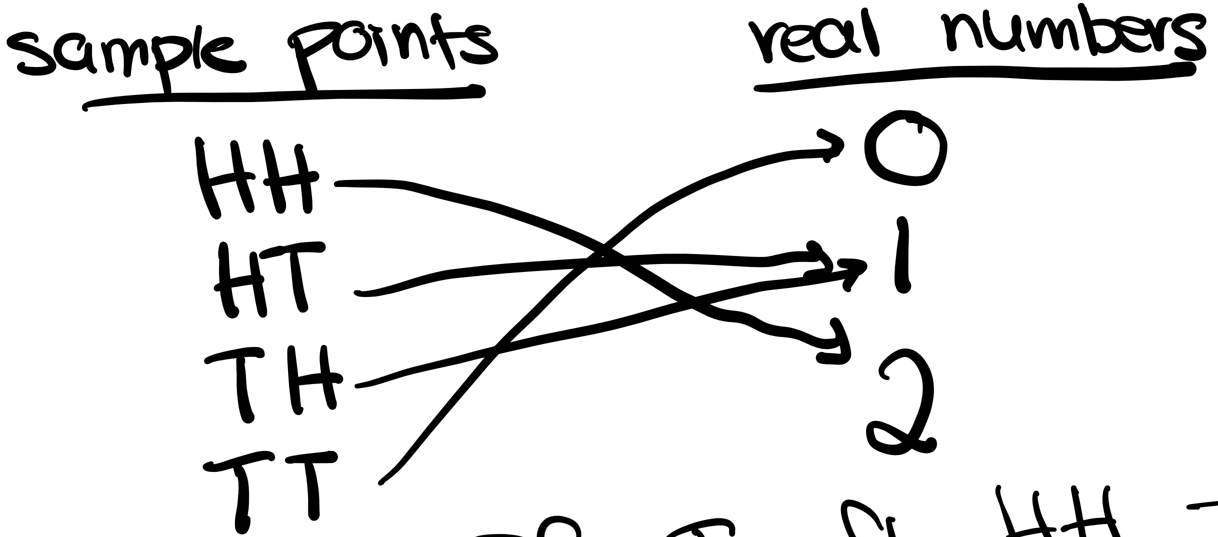
sample points	real numbers
$w_1 \xrightarrow{X(w_1)=4} 4$	-1
$w_2 \to 7$	4
$\vdots$	7
$w_n \to 4$	22.5
	$\vdots$

Whenever my outcome is w_1 or w_n, the value of X will be 4.

Example

$P(\text{heads}) = \frac{1}{2}$

We have a fair coin. Let X be a random variable corresponding to a series of two coin flips, where X is equal to the number of heads in the two coin flips.



If I flip HH, then $X=2$

Random Variables (cont.)

Events: X = a → the event that some sample point that maps to a is the outcome of our experiment.

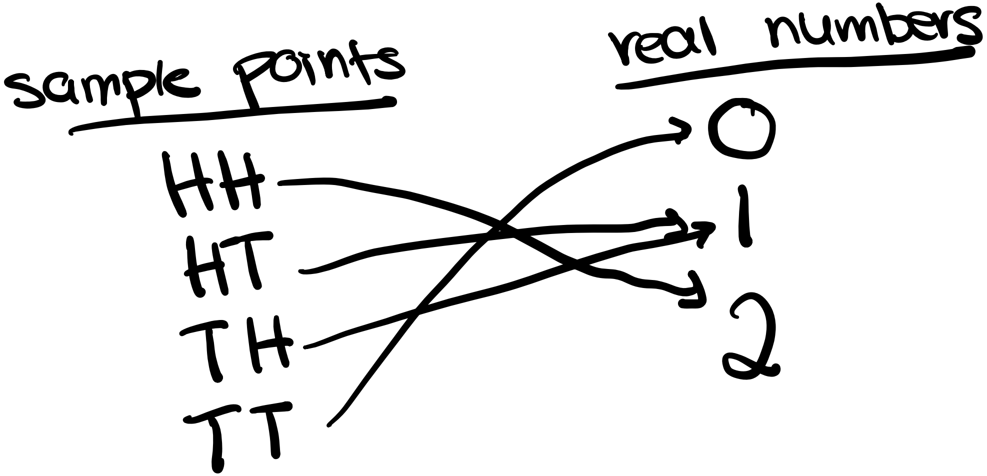
$$P(A) = \sum_{w \in A} P(w)$$

$$\{w \mid X(w) = a\}$$

$$P(X=a) = \sum_{\{w \mid X(w)=a\}} P(w)$$

Example

We have a fair coin. Let X be a random variable corresponding to a series of two coin flips, where X is equal to the number of heads in the two coin flips.



$$P(X=0) = P(TT) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(X=1) = P(HT) + P(TH) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

$$P(X=2) = P(HH) = \frac{1}{2} \cdot \frac{1}{2}$$

Random Variables (cont.)

Definition 15.2 (Distribution)**.** *The* <u>*distribution*</u> *of a discrete random variable X is the collection of values $\{(a, \mathbb{P}[X = a]) : a \in \mathscr{A}\}$, where $\mathscr{A}$ is the set of all possible values taken by X.*

X: the RV corresponding to # of heads

dist. : $\begin{cases} 0 & \text{w/ prob } \frac{1}{4} \\ 1 & \text{w/ prob } \frac{1}{2} \\ 2 & \text{w/ prob } \frac{1}{4} \end{cases}$

Note: events $X = k_1$ and $X = k_2$ are **disjoint**: no sample point maps to two different values

- The union of all events is the **sample space,** the probabilities of all the events must add to 1.

Joint Distributions for two Random Variables

Definition 15.3. *The joint distribution for two discrete random variables X and Y is the collection of values* $\{((a,b),\mathbb{P}[X=a,Y=b]) : a \in \mathscr{A},\ b \in \mathscr{B}\}$, *where* $\mathscr{A}$ *is the set of all possible values taken by X and* $\mathscr{B}$ *is the set of all possible values taken by Y.*

Example: two dice rolls!

X: outcome of dice roll 1

Y: outcome of dice roll 2

	probabilities
(1,1)	1/36
→ (1,2)	1/36
⋮	⋮
(6,6)	1/36

Joint Distributions for two Random Variables

The **marginal distribution** of X (visualization!):

X	Y	$P(X=x, Y=y)$
1	2	1/5
1	3	1/5
2	2	1/5
3	3	1/5
4	5	1/5

$P(X=1)$

$= 2/5$

Joint Distributions for Two Random Variables

$$\mathbb{P}[X = a] = \sum_{b \in \mathscr{B}} \mathbb{P}[X = a, Y = b].$$

Marginal distribution

Independence of Random Variables

Two random variables are **independent** if:

$$\boxed{P(X=a, Y=b) = P(X=a)P(Y=b)}$$

idea of random variables

$$P(A \cap B) = P(A)P(B)$$

$$P(X=3, Y=4) = P(X=3)P(Y=4) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

(or <u>Indicator</u>)

Bernoulli Random Variables

The <u>Bernoulli distribution</u> of a random variable is **defined as**:

$X \sim \text{Bernoulli}(p) \qquad 0 \leq p \leq 1$

$$X\text{'s distribution}: \begin{cases} 1 & \text{w/ probability } p \\ 0 & \text{w/ probability } 1-p \end{cases}$$

Example Bernoulli

Example: Coin Flip with a coin where P(heads) = 1/3

X: represents outcome of the coin flip

↳ X is 1 when we flip a head

↳ 0 otherwise

$$X\text{'s distribution: } \begin{cases} 1 & \text{w/ prob } \frac{1}{3} \\ 0 & \text{w/ prob } \frac{2}{3} \end{cases}$$

$X \sim \text{Bernoulli}(\frac{1}{3})$ distribution

"Indicator" : indicates if something has happened

Example Bernoulli

Example: Dice rolling an even number

X: did dice roll even

$$X: \begin{cases} 1 & \text{w/ prob } \frac{1}{2} \\ 0 & \text{w/ prob } \frac{1}{2} \end{cases}$$

$$X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$$

A sum of Bernoulli RVs: Binomial

What if we have a sum of 10 independent coin flips, where each has probability ⅓ of landing heads? Let X be the number of heads total.

- What is $P(X = 5)$?

→ 5 of the X_is landed heads
5 of them landed tails

X: number of heads total

X_i: 1 if flip i landed heads
0 otherwise

$X = X_1 + X_2 + X_3 + \dots X_{10}$

$X_i \sim \text{Bernoulli}(\frac{1}{3})$

each one heads w/ prob $\frac{1}{3}$

$$\boxed{\binom{10}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^5 = P(X=5)}$$

prob 5 land heads?

Binomial (cont.)

More generally, a **binomial random variable** is a sum of n bernoulli random variables. Parameters:

1. n : number of bernoulli RVs you're summing (10)
2. p : probability of success for each bernoulli RV $\left(\frac{1}{3}\right)$

P(X = k) = $\binom{n}{k} p^k (1-p)^{n-k}$

$0 \leq k \leq n$

probabilities

values

Example Binomial

You roll 5 dice. What's the probability that you rolled 3 6s?

X: number of 6s that I roll

X_i: $\begin{cases} 1 & \text{if roll 6} \\ 0 & \text{otherwise} \end{cases}$

$X = X_1 + X_2 + X_3 + \dots X_5$

$X_i \sim \text{Bernoulli}\left(\frac{1}{6}\right)$

$X \sim \text{Binomial}\left(5, \frac{1}{6}\right)$

$P(X=3) = \binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$

Example Binomial

Every day, Sodoi Coffee opens at 10am with probability 80%. What is the probability Sodoi Coffee is open at 10am for four days this week?

X: number of days that Sodoi opens at 10am

$X = X_1 + X_2 + \dots X_7$ $\quad X_i : \begin{cases} 1 & \text{if Sodoi opens at 10am that day} \\ 0 & \text{otherwise} \end{cases}$

$X \sim \text{Binomial}(7, 0.8)$ $\quad X_i \sim \text{Bernoulli}(0.8)$

$P(X=4) = \binom{7}{4}(0.8)^4(0.2)^3$

Binomial Example: Pulling balls out of a bag

Let's say there are **r** red balls and **g** green balls in a bag. If I pick from the bag 5 times **with replacement**, what's the probability I get exactly ~~two red balls~~ three green balls?

X: number of green balls

$X = X_1 + X_2 + \dots X_5$

$X_i : \begin{cases} 1 & \text{if I get a green ball} \\ 0 & \text{otherwise} \end{cases}$

$X \sim \text{Binomial}\left(5, \frac{g}{r+g}\right)$

$X_i \sim \text{Bernoulli}\left(\frac{g}{r+g}\right)$

$P(X=3) = \binom{5}{3}\left(\frac{g}{r+g}\right)^3\left(\frac{r}{r+g}\right)^2$

Hypergeometric: What if I pick without replacement?

r red balls, g green balls
r+g total

What is the probability of getting two red balls, and then three green balls? What about RGRGG?

$$\frac{r}{r+g} \times \frac{r-1}{r+g-1} \times \frac{g}{r+g-2} \times \frac{g-1}{r+g-3} \times \frac{g-2}{r+g-4} \quad RRGGG$$

$$RGRGG: \frac{r}{r+g} \times \frac{g}{r+g-1} \times \frac{r-1}{r+g-2} \times \frac{g-1}{r+g-3} \times \frac{g-2}{r+g-4}$$

same

Hypergeometric: What if I pick without replacement?

What is the probability of getting a **specific sequence** of three green balls? What is the probability of getting three green balls in general?

$$\frac{r \times r-1 \times g \times g-1 \times g-2}{(r+g) \times (r+g-1) \times (r+g-2) \times (r+g-3) \times (r+g-4)}$$

$$\frac{r!}{(r-2)!} \cdot \frac{g!}{(g-3)!} \cdot \frac{(r+g-5)!}{(r+g)!} \cdot \frac{5!}{3!\,2!}$$

$$\frac{r!}{(r-2)!\,2!} \cdot \frac{g!}{(g-3)!\,3!} \cdot \frac{(r+g-5)!\,5!}{(r+g)!}$$

$$P(X=3) = \frac{\binom{r}{2}\binom{g}{3}}{\binom{r+g}{5}}$$

hypergeometric

$$\binom{r+g}{5} = \frac{r+g!}{(r+g-5)!\,5!}$$

Hypergeometric (cont.)

total # of elements

# of good elements

Hypergeometric: X ~ Hypergeometric(N, G, n)

number of elements you sample without replacement

$$P(X=k) = \frac{\binom{G}{k}\binom{N-G}{n-k}}{\binom{N}{n}}$$

"bad elements"

n-k "bad things"

$$0 \le k \le n$$

Hypergeometric (cont.)

$N: 13 \quad G: 3$

$n: 3$

There are 13 Taylor Swift albums, three of which are "Taylor's Version". If I randomly sample three albums **without replacement** what's the chance that exactly one of them is "Taylor's Version"?

X: # of Taylor's Version albums I draw

$$P(X=1) = \frac{\binom{3}{1}\binom{13-3}{2}}{\binom{13}{3}}$$

Poisson Random Variables

The Poisson distribution models **rare events** where the average number of occurrences of some event in a unit of time is λ.

Definition 19.2 (Poisson distribution). *A random variable X for which*

$$\mathbb{P}[X = i] = \frac{\lambda^i}{i!} e^{-\lambda}, \qquad \text{for } i = 0, 1, 2, \ldots$$

could be infinite number of occurences

T

is said to have the Poisson distribution with parameter λ. This is abbreviated as $X \sim \text{Poisson}(\lambda)$.

Poisson (cont.)

$\lambda = 1$

Taylor Swift releases an average of one album a year. What's the probability she releases 2 albums this year?

$i=2$

$$P(X=i) = \frac{\lambda^i}{i!} e^{-\lambda}$$

X: # of albums Taylor released this year

$$P(X=2) = \frac{1^2}{2!} e^{-1}$$

Poisson & Binomial Connection

Let X ~ Binomial(n, λ / n). What is P(X = k) as n goes to infinity?

$$P(X=k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n!}{(n-k)!\,k!}\,\frac{\lambda^k}{n^k}\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!}\left(\frac{n!}{(n-k)!\,n^k}\right)\left(1-\frac{\lambda}{n}\right)^{n-k}$$

Poisson & Binomial Connection

as $n \to \infty$

$$\frac{\cancel{n^k + bn^{k-1} + \dots 1}}{n^k} \leftarrow \frac{n(n-1)\dots(n-k+1)}{n^k} = \cancel{\frac{an^k + \dots \dots 0}{n^k}} \to 1$$

$$= \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k}$$

$1+x \sim e^x$ $\approx \frac{\lambda^k}{k!}\left(e^{-\lambda/n}\right)^n\left(e^{-\frac{\lambda}{n}}\right)^{-k}$

$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\dots$

$\approx \frac{\lambda^k}{k!}e^{-\lambda}\cancel{\left(e^{-\frac{\lambda k}{n}}\right)}$

$= \boxed{e^{-\lambda}\frac{\lambda^k}{k!}}$

as $n \to \infty$

$\frac{\lambda k}{n} \to 0$

$e^{-\frac{\lambda k}{n}} \to e^{-0} \to$

$e^0 = 1$

Sum of Independent Poisson Random Variables

Theorem 19.5. *Let $X \sim$ Poisson(λ) and $Y \sim$ Poisson(μ) be independent Poisson random variables. Then, $X+Y \sim$ Poisson$(\lambda+\mu)$.*

$$P(X+Y=n) = \sum_{k=0}^{n} P(X=k, Y=n-k)$$

$$= \sum_{k=0}^{n} P(X=k)P(Y=n-k)$$

$$= \sum_{k=0}^{n} \frac{\lambda^k}{k!} e^{-\lambda} \frac{\mu^{n-k}}{(n-k)!} e^{-\mu}$$

Sum of Independent Poisson Random Variables

$$e^{-(\mu+\lambda)} \sum_{k=0}^{n} \frac{\lambda^k \mu^{n-k}}{k!(n-k)!}$$

$$\frac{n!}{n!}$$

$$\binom{n}{k}$$

$$\frac{e^{-(\mu+\lambda)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda^k \mu^{n-k}$$

Sum of Independent Poisson Random Variables

Theorem 10.1 (Binomial Theorem). *For all $n \in \mathbb{N}$,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

$$\frac{e^{-(\mu+\lambda)}}{n!} (\lambda+\mu)^n \leftarrow \text{Poisson}(\lambda+\mu) = n$$

Geometric Random Variables

Another cool application is **geometric random variables.** Let's say I flip a coin that lands heads with probability **p** repeatedly until I get a head.

What's the probability it takes 5 flips to get a head?

$\underline{T}\ \underline{T}\ \underline{T}\ \underline{T}\ \underline{H}$

$P(\text{it takes 5 flips}) = (1-p)^4 p$

Geometric Random Variables (cont.)

Formally, a random variable with parameter **p** where **p** is the probability of success. Continue repeatedly until you get a success.

fail
k-1 times

→ every trial has probability p of succeeding

Let X ~ Geometric(p)

P(X = k) = $\underbrace{(1-p)^{k-1}}_{}$ p ← succeed on your last try

Example: Roll a dice <u>until</u> I get a 6. Probability it takes me 3 dice rolls?

$P(\text{success}) = \frac{1}{6}$ $\qquad X \sim \text{Geometric}\left(\frac{1}{6}\right)$

$$P(X=3) = \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)$$

Geometric Random Variables (cont.)

Check that it's a distribution!

Sum of infinite geometric sequence is,

$$a + ar^2 + ar^3 + \ldots. = \boxed{\frac{a}{1-r}}, \text{ when } r < 1$$

Memorylessness of Geometric RVs

A really cool property of geometric random variables

- Let's say I have a coin that flips heads with probability **p.** If I flip 10 tails in a row, what's the probability that I will get a head in 12 total flips?

on 12th flip

$$P(\text{head in 12 flips} \mid \text{10 tails}) = \frac{P(\text{10 tails} \wedge \text{head in 12 flips})}{P(\text{10 tails})}$$

T T T T T T T T T T T H

$$\frac{(1-p)^{11}p}{(1-p)^{10}} = (1-p)p$$

succeeds

$(1-p)$

on the 2nd flip

Memorylessness of Geometric RVs

Recap

- Learned about the definitions of Random Variables
  - Independence of Random Variables
  - Joint Distributions of Random Variables

- Types of RVs & their relations to each other
  - Bernoulli, Binomial, Hypergeometric, Poisson
    - Sum of independent Poissons is **also** a Poisson
  - Geometric
    - Memorylessness Property of Geometric RVs

When to use each RV :)