Lecture 4D: Random Variables

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What is a random variable?

Definition 15.1 (Random Variable). *A random variable X on a sample space Ω is a function $X : \Omega \to \mathbb{R}$ that assigns to each sample point $\omega \in \Omega$ a real number $X(\omega)$.*

Example

We have a fair coin. Let X be a random variable corresponding to a series of two coin flips, where X is equal to the number of heads in the two coin flips.

Random Variables (cont.)

Events: $X = a$

Example

We have a fair coin. Let X be a random variable corresponding to a series of two coin flips, where X is equal to the number of heads in the two coin flips.

Random Variables (cont.)

Definition 15.2 (Distribution). *The distribution of a discrete random variable X is the collection of values $\{(a, \mathbb{P}[X = a]) : a \in \mathscr{A}\}$, where $\mathscr{A}$ is the set of all possible values taken by X.*

Note: events $X = k_1$ and $X = k_2$ are **disjoint**

- The union of all events is the **sample space,** the probabilities of all the events must add to 1.

Joint Distributions for two Random Variables

Definition 15.3. *The joint distribution for two discrete random variables X and Y is the collection of values* $\{((a,b),\mathbb{P}[X=a,Y=b]) : a \in \mathscr{A},\ b \in \mathscr{B}\}$, *where* $\mathscr{A}$ *is the set of all possible values taken by X and* $\mathscr{B}$ *is the set of all possible values taken by Y.*

Example: two dice rolls!

Joint Distributions for two Random Variables

The **marginal distribution** of X (visualization!):

Joint Distributions for Two Random Variables

$$\mathbb{P}[X = a] = \sum_{b \in \mathscr{B}} \mathbb{P}[X = a, Y = b].$$

Independence of Random Variables

Two random variables are **independent** if:

Bernoulli Random Variables

The Bernoulli distribution of a random variable is **defined as**:

Example Bernoulli

Example: Coin Flip with a coin where P(heads) = 1/3

Example Bernoulli

Example: Dice rolling an even number

A sum of Bernoulli RVs: Binomial

What if we have a sum of 10 independent coin flips, where each has probability ⅓ of landing heads? Let X be the number of heads total.

- What is $P(X = 5)$?

Binomial (cont.)

More generally, a **binomial random variable** is a sum of n bernoulli random variables. Parameters:

1.
2.

P(X = k) =

Example Binomial

You roll 5 dice. What's the probability that you rolled 3 6s?

Example Binomial

Every day, Sodoi Coffee opens at 10am with probability 80%. What is the probability Sodoi Coffee is open at 10am for four days this week?

Binomial Example: Pulling balls out of a bag

Let's say there are **r** red balls and **g** green balls in a bag. If I pick from the bag 5 times **with replacement**, what's the probability I get exactly two red balls?

Hypergeometric: What if I pick without replacement?

What is the probability of getting two red balls, and then three green balls? What about RGRGG?

Hypergeometric: What if I pick without replacement?

What is the probability of getting a **specific sequence** of three green balls? What is the probability of getting three green balls in general?

Hypergeometric (cont.)

Hypergeometric: $X \sim \text{Hypergeometric}(N, G, n)$

Hypergeometric (cont.)

There are 13 Taylor Swift albums, three of which are "Taylor's Version". If I randomly sample three albums **without replacement** what's the chance that exactly one of them is "Taylor's Version"?

Poisson Random Variables

The Poisson distribution models **rare events** where the average number of occurrences of some event in a unit of time is λ.

Definition 19.2 (Poisson distribution). *A random variable X for which*

$$\mathbb{P}[X = i] = \frac{\lambda^i}{i!} e^{-\lambda}, \qquad \text{for } i = 0, 1, 2, \ldots$$

is said to have the Poisson distribution with parameter λ. This is abbreviated as $X \sim \text{Poisson}(\lambda)$.

Poisson (cont.)

Taylor Swift releases an average of one album a year. What's the probability she releases 2 albums this year?

Poisson & Binomial Connection

Let $X \sim \text{Binomial}(n, \lambda / n)$. What is $P(X = k)$ as n goes to infinity?

Poisson & Binomial Connection

Sum of Independent Poisson Random Variables

Theorem 19.5. *Let* $X \sim \text{Poisson}(\lambda)$ *and* $Y \sim \text{Poisson}(\mu)$ *be independent Poisson random variables. Then,* $X + Y \sim \text{Poisson}(\lambda + \mu)$.

Sum of Independent Poisson Random Variables

Sum of Independent Poisson Random Variables

Theorem 10.1 (Binomial Theorem). *For all $n \in \mathbb{N}$,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Geometric Random Variables

Another cool application is **geometric random variables.** Let's say I flip a coin that lands heads with probability **p** repeatedly until I get a head.

What's the probability it takes 5 flips to get a head?

Geometric Random Variables (cont.)

Formally, a random variable with parameter **p** where **p** is the probability of success. Continue repeatedly until you get a success.

Let X ~ Geometric(p)

P(X = k) =

Geometric Random Variables (cont.)

Check that it's a distribution!

Sum of infinite geometric sequence is,

$$a + ar^2 + ar^3 + \dots = \boxed{\frac{a}{1 - r}}, \text{ when } r < 1$$

Memorylessness of Geometric RVs

A really cool property of geometric random variables

- Let's say I have a coin that flips heads with probability **p.** If I flip 10 tails in a row, what's the probability that I will get a head in 12 total flips?

Memorylessness of Geometric RVs

Recap

- Learned about the definitions of Random Variables
    - Independence of Random Variables
    - Joint Distributions of Random Variables

- Types of RVs & their relations to each other
    - Bernoulli, Binomial, Hypergeometric, Poisson
        - Sum of independent Poissons is **also** a Poisson
    - Geometric
        - Memorylessness Property of Geometric RVs