Announcements:

↳ Course Evals (> 80% of the class fills out the eval, 1 point of EC post-curve)

→ HW 7 (next week's hw) released on Friday

↳ due a day earlier

↳ 3 questions long

Lecture 6A:

Markov Chains

→ 12 days

UC Berkeley CS70

Summer 2023

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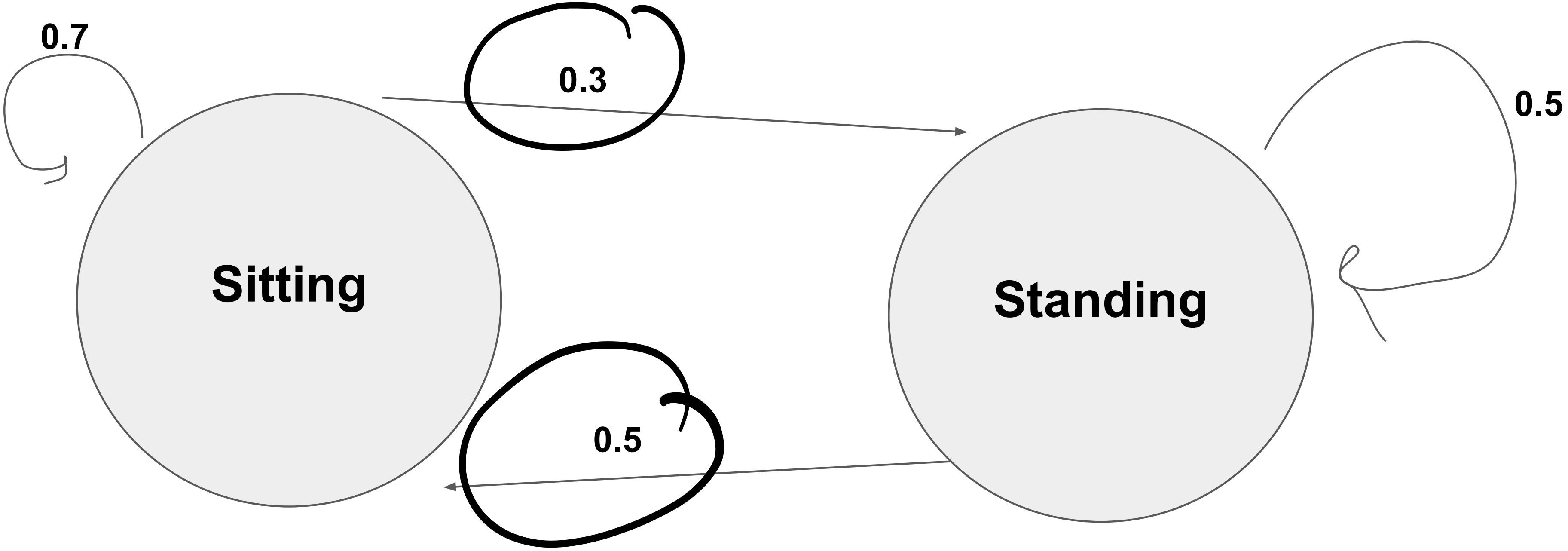
Finite Markov Chains

Markov Chains help us to visualize **transitions** between different **states.**



Finite Markov Chains (Formally)

Markov Chains are defined by two key components:

- State Space – set of all possible **states** that your experiment can exist in (all possible values of the random variable in the Markov Chain). i.e. {sitting, standing}.

Finite Markov Chains (Formally)

probability of doing that

Markov Chains are defined by two key components:

- Transition Probability Matrix – the **probability** of moving from any **one state** to any **other state**

state you're moving to

$$
\begin{array}{c|cc}
 & \text{Sit} & \text{Stand} \\
\hline
\text{Sit} & 0.7 & 0.3 \\
\text{Stand} & 0.5 & 0.5
\end{array}
$$

state you're on

Key Information!

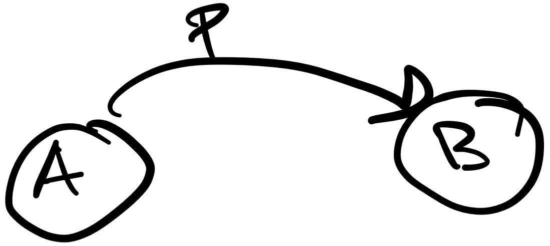
The **most** important thing about Markov chains, is that the chain is **amnesic (memoryless)** meaning each step should **only** depend on the previous state.

X_i : State that I'm in at step i

$\{A, B, C\}$

$$P(X_n = A \mid X_{n-1} = B, X_{n-2} = A, X_{n-3} = C \ldots)$$

$$= P(X_n = A \mid X_{n-1} = B)$$



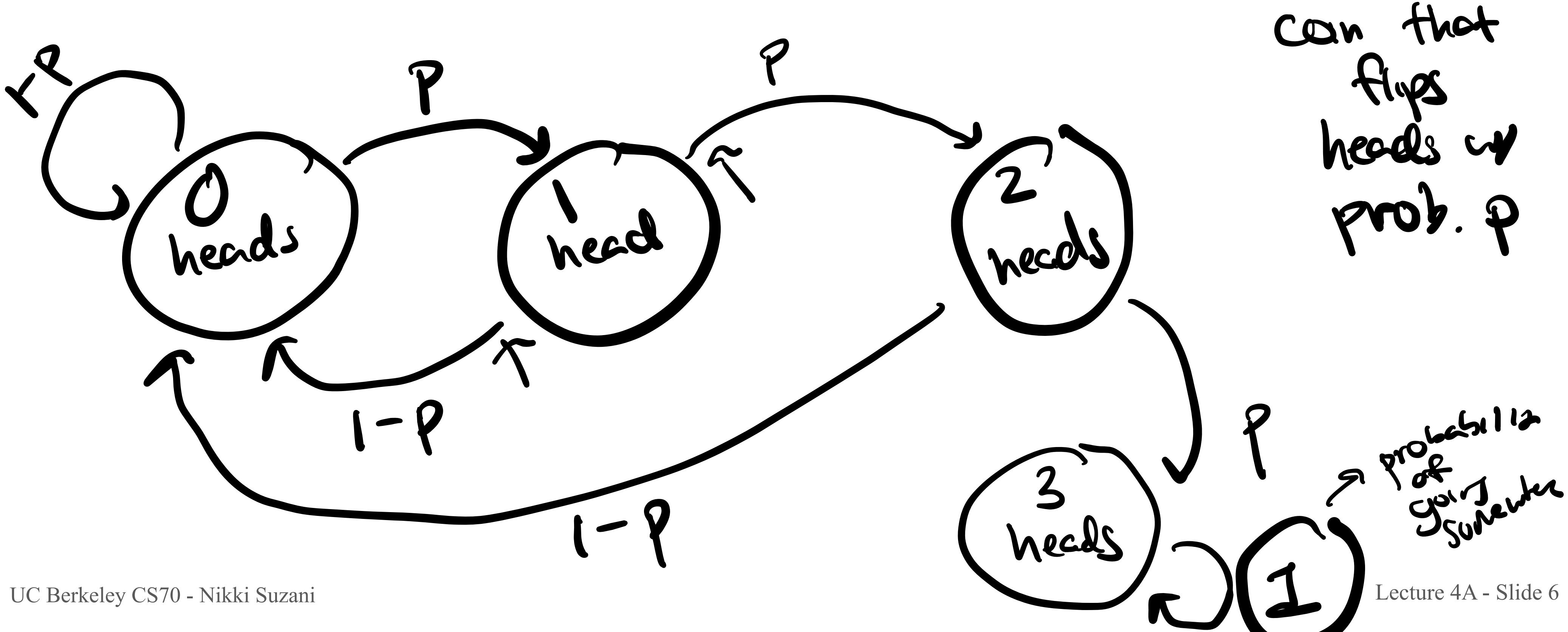
HHT

1-p+p

Can the following be modeled as a Markov Chain?

three

You have a coin that you flip continuously until you get ~~two~~ **consecutive** heads.



Modeled as a Markov Chain?

You're drawing **without replacement** from a bag with a purple, red, and green ball. Model the outcomes of your draws.

$$RGB \xrightarrow{1/3} RG, \quad RG \to R, \quad RG \to G$$

purple ball
draw 1

green ball
draw 2

100% prob. that I got a red ball
draw 3

50% prob. of red & a 50% prob. of purple

$\varphi \to$

More Vocab!

$[0.3 \quad 0.4 \quad 0.3]$

A

Often for Markov Chain questions you're given an **initial distribution**. This tells you the probability of being in **each state** when you start. Each probability should be >= 0, and they should sum to 1.

in state 1

in state n

$$\pi = [\pi(0) \quad \pi(1) \quad \cdots \quad \pi(n)]$$

probability of starting in state 0

$$0 \le \pi(i) \le 1$$

$$\pi(0) + \ldots \pi(n) = 1$$

Using the initial distribution

For some Markov Chain, let π = [0.3 0.4 ~~0.2 0.1~~] for states A, B, and C respectively.

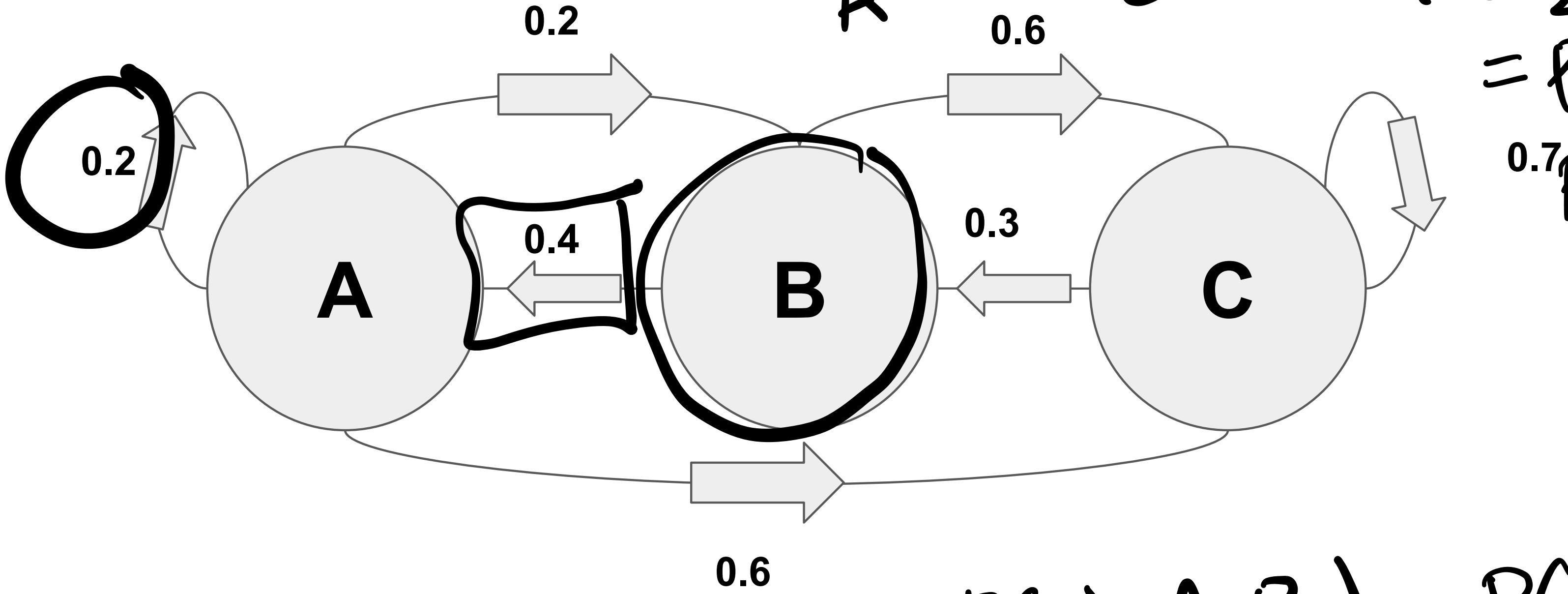
0.3

A B C

$P(X_2 = A \mid X_0 = B)$

$= P(X_0 = B) P(X_1 = A \mid X_0 = B)$

$P(X_2 = A \mid X_1 = a)$



$P(A) = P(A \cap B) + P(A \cap \overline{B})$

Using the initial distribution

0.3

Let π = [0.3 0.4 ~~0.2 0.1~~]. After **one step**, what is the probability of being in **state A?**

T

Initial dist : [0.3 0.4 0.3]

Three States: A, B, C

Starting in B: P(state A): P(B → A) = 0.4

$$P(X_1 = A) = P(X_1 = A \mid X_0 = A)\,\boxed{P(X_0 = A)}$$

$$+ \; P(X_1 = A \mid X_0 = B)\,P(X_0 = B)$$

$$+ \; P(\;\; = A \mid X_0 = C)\,P(X_0 = C)$$

$$\begin{bmatrix} P(X_0=A) & P(X_0=B) & P(X_0=C) \end{bmatrix}$$

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & P(X_1=A\mid X_0=A) & & \\ B & P(X_1=A\mid X_0=B) & & \\ C & P(X_1=A\mid X_0=C) & & \end{array}$$

$$P(X_1=A) = P(X_0=A)P(X_1=A\mid X_0=A) + P(X_0=B)P(X_1=A\mid X_0=B)$$

after step 1 vector $= \pi P$

initial dist.

probability matrix

after step 2 vector $=$ After step 1 vector $\cdot P$

$= \pi P^2$

[N(A) N(B) N(C)]

after step 3 $= \pi P^3$

$$\boxed{\text{after } n \text{ steps} = \pi P^n}$$

probability of state A after n steps

1st entry of this new vector

$= 0.2 \times 0.\bar{3} + 0.4 \times 0.4 + 0 \cdot \_$

$= 0.2 \times 0.3 + 0.4 \times 0.4 \leftarrow$

$= 0.16 + 0.06 = 0.22$

$P(X_1 = B) = 0.48$

$P(X_1 = C) = 0.3$

Probability of where you are after step 1

is $[\boxed{0.22} \; 0.48 \; 0.3]$

new <u>initial</u> dist

Generalize!

Let's look at the new matrix that we make. Now, what's the probability of being in State A after **two steps**?

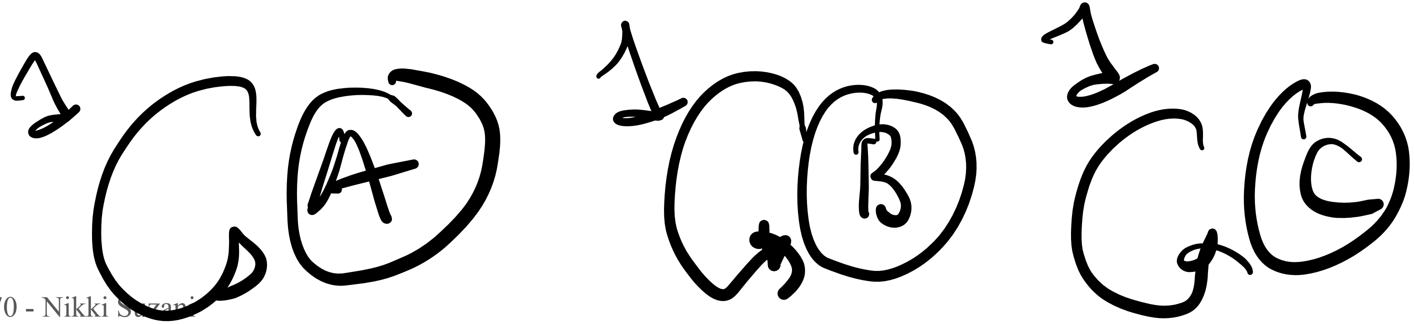
$$P(X_2 = A) = P(X_2 = A \mid X_1 = A) P(X_1 = A)$$

$$+ \ldots\ldots$$

Generalize

How do we find the probability of being in a certain state after **n steps**?

$$\pi_n = \pi P^n \quad \left(\begin{matrix}\text{1st} \\ \text{entry}\end{matrix}\right)$$



More Vocab!

$[0.3 \quad 0.4 \quad 0.3]$

$[1 \quad 0 \quad 0]$

An **invariant (or stationary)** distribution π is a starting distribution such that:

- π = πP

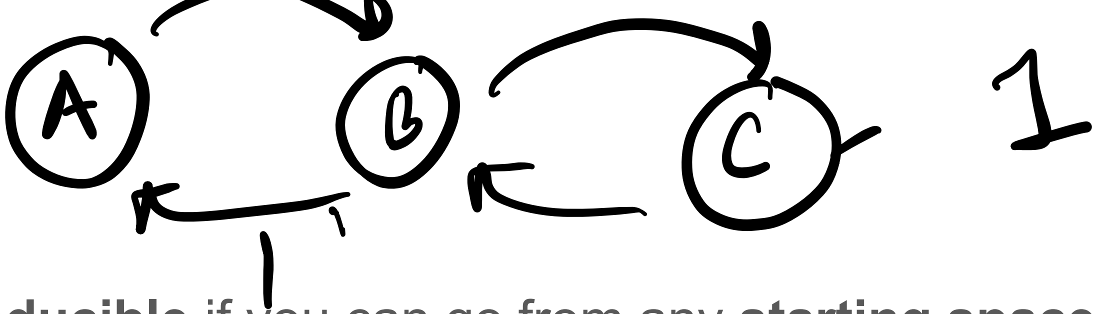
when I multiply my initial dist. by my probability matrix, I get the same distribution

example invariant

$$[a \quad b \quad c] \begin{array}{c|ccc} & A & B & C \\ \hline A & 1 & 0 & 0 \\ B & 0 & 1 & 0 \\ C & 0 & 0 & 1 \end{array} = [a \quad b \quad c]$$

1

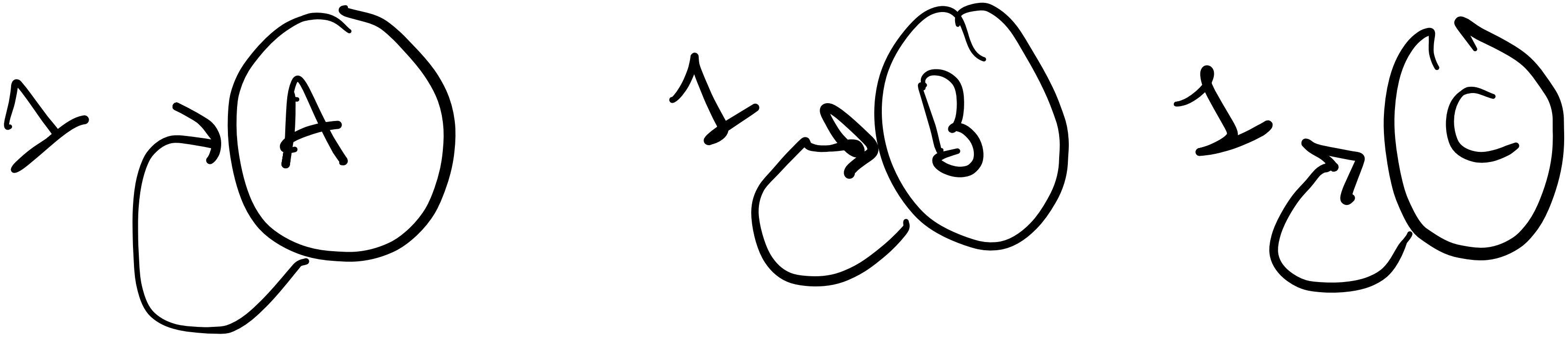
More Vocab!



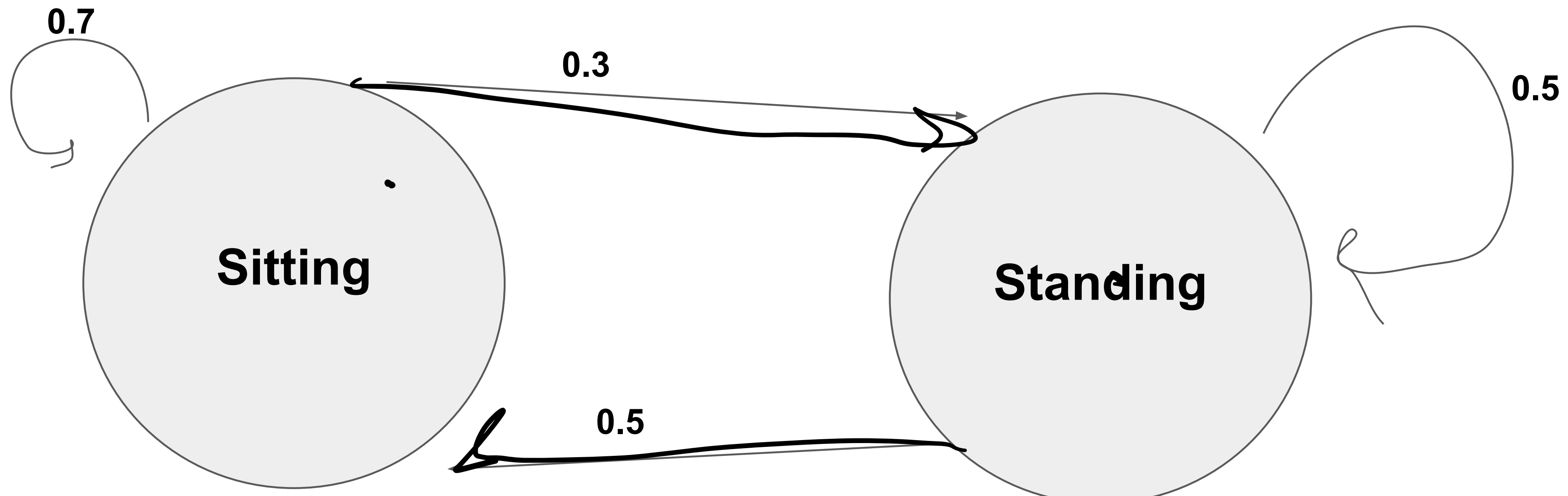
A Markov chain is said to be **irreducible** if you can go from any **starting space** to any other **ending space**.

- Specifically, there is a path of nonzero (positive) probability that goes from state **i** to state **j** for all i & j.

not irreducible

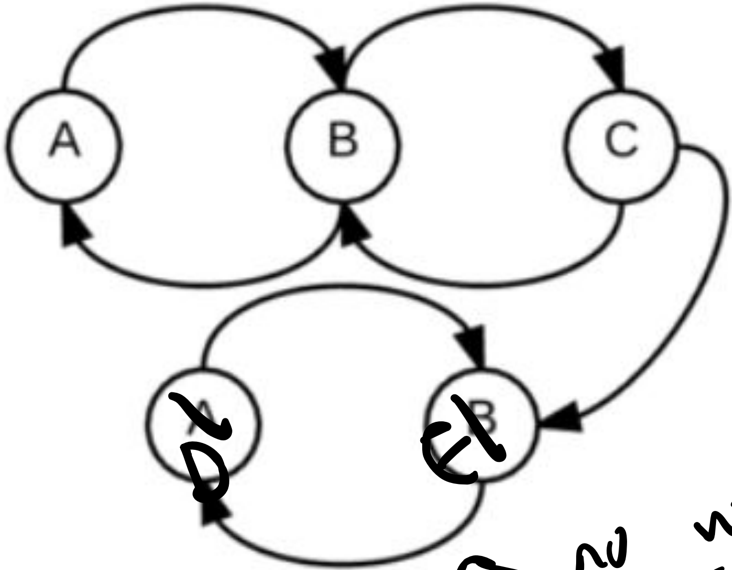


Irreducible or Not?



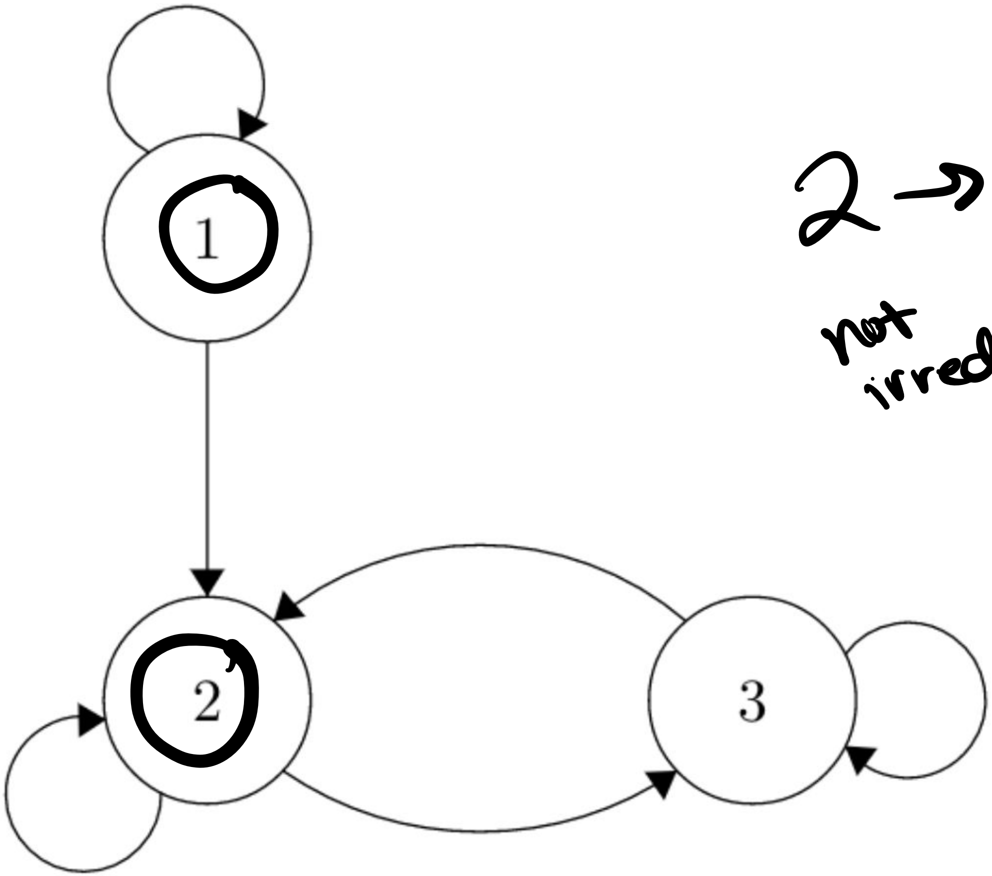
Irreducible or Not?

not irreducible



no way out

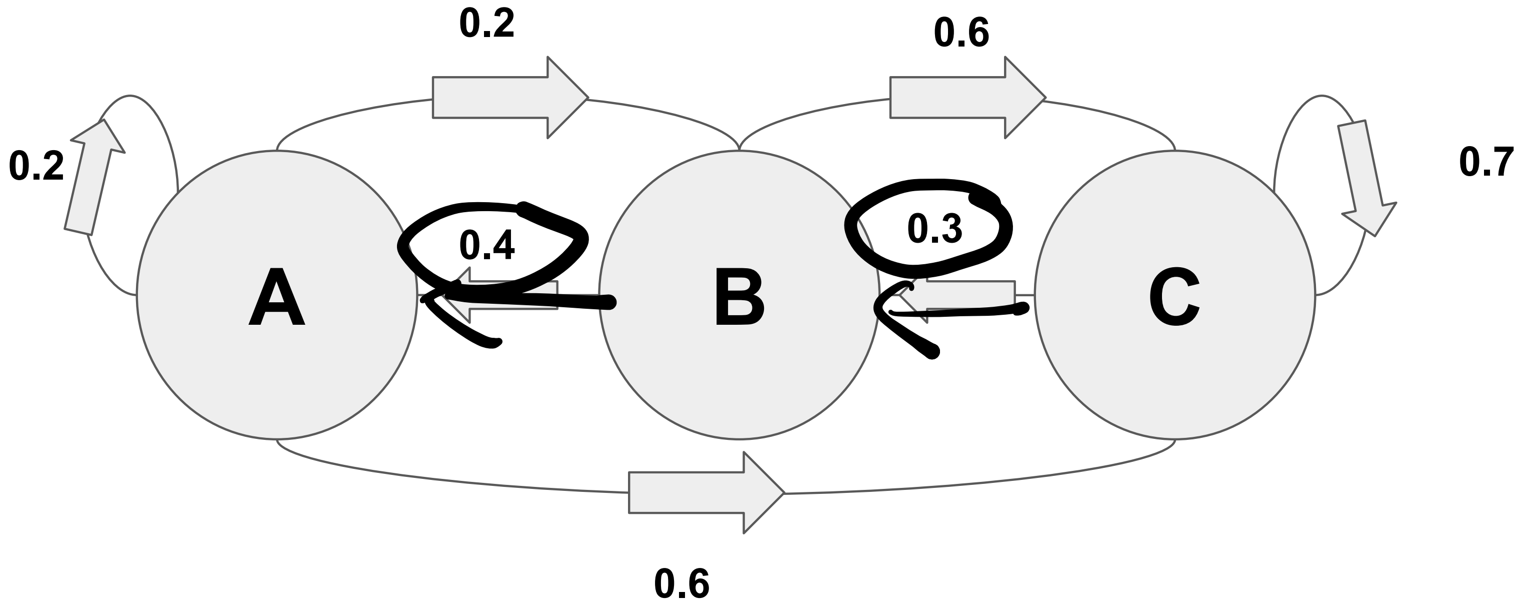
Irreducible or Not?

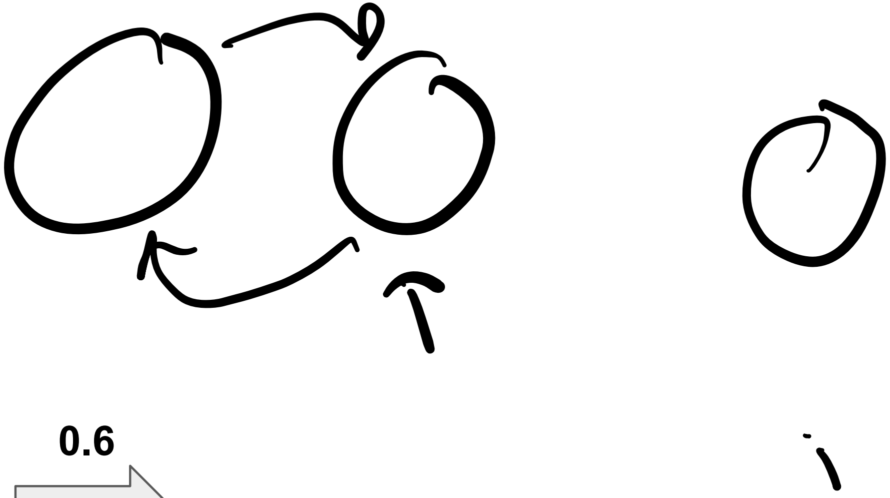


2 → 1

not irreducible

Irreducible or Not?

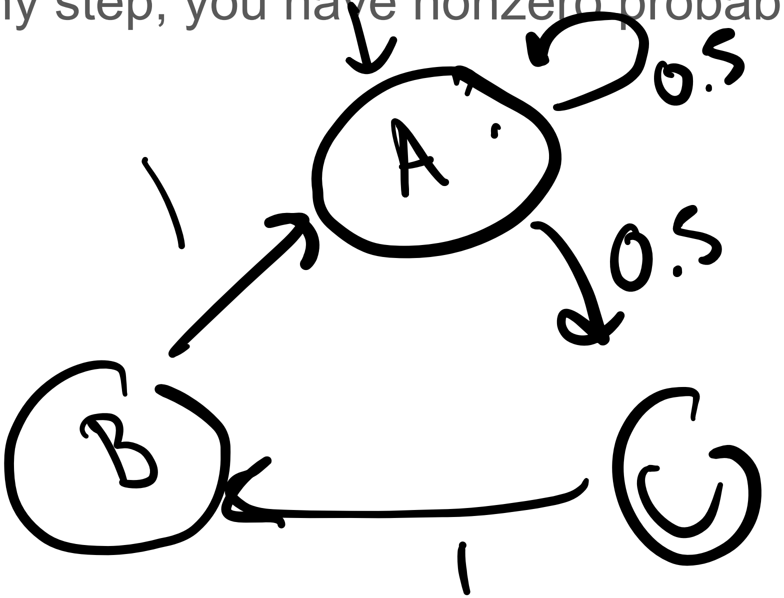




More Vocab!

A **state** i has **period** d if the Markov Chain can only come back to **i** in multiples of **d.**

- A Markov Chain is said to be **aperiodic** if the period of the chain is 1 – meaning at any step, you have nonzero probability of being in any state.



Aperiodic (Formally)

Theorem 22.4. *Consider an irreducible Markov chain on $\mathscr{X}$ with transition probability matrix P. Define*

$$d(i) := g.c.d\{n > 0 \mid P^n(i,i) = Pr[X_n = i|X_0 = i] > 0\}, i \in \mathscr{X}. \tag{16}$$

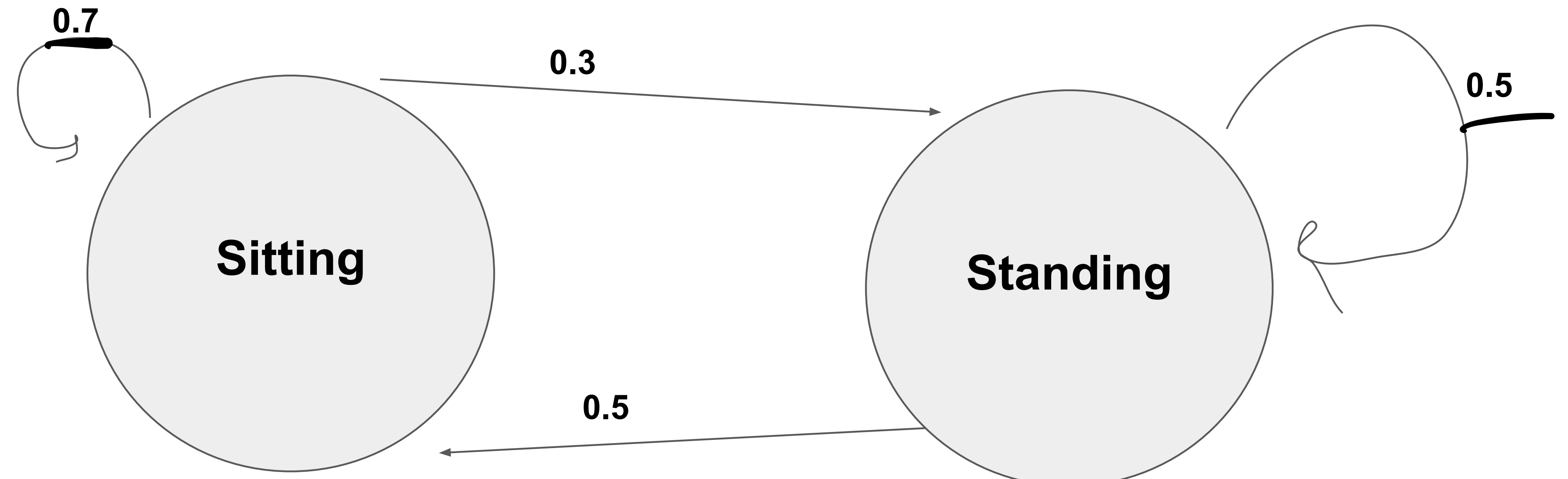
(a) Then, $d(i)$ has the same value for all $i \in \mathscr{X}$. If that value is 1, *the Markov chain is said to be* aperiodic. *Otherwise, it is said to be* periodic with period d.

Aperiodic Quick Tip!

If any state in an irreducible Markov Chain has a **self-loop with nonzero probability**, then it is **aperiodic**.

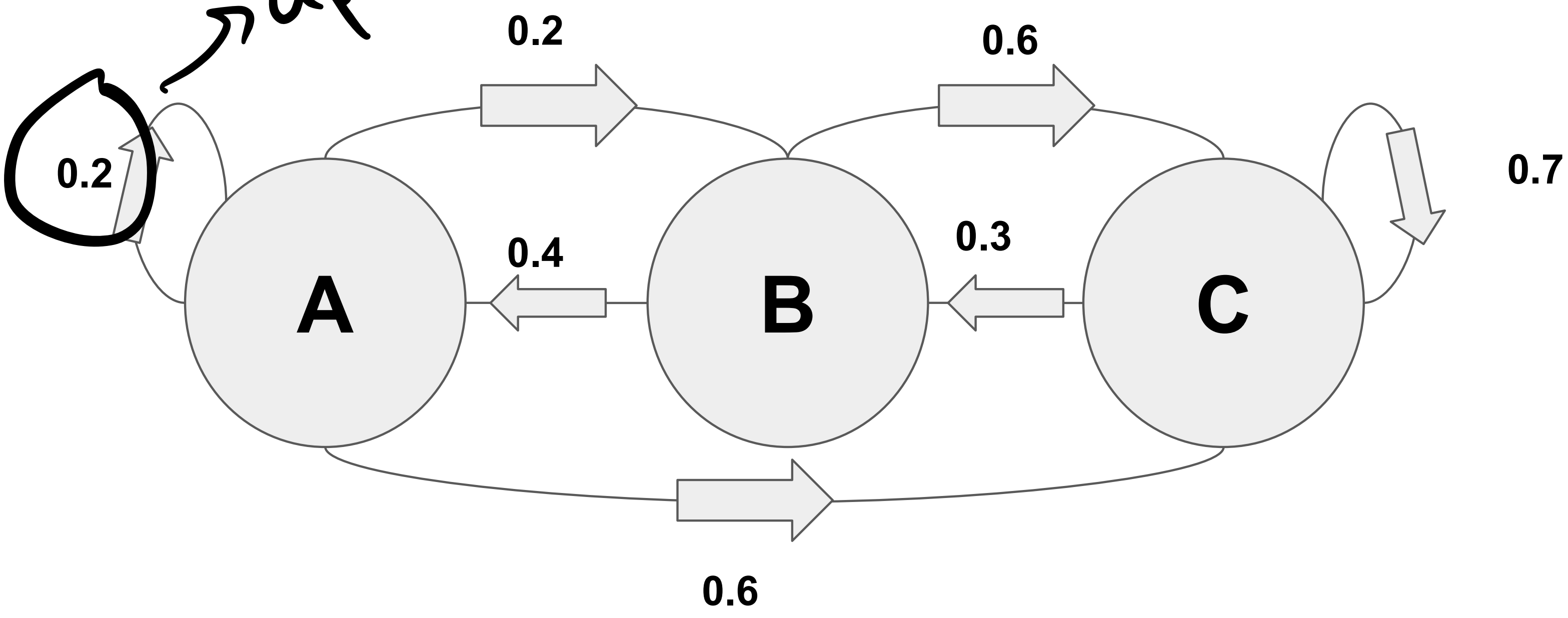
Aperiodic or Not?

Aperiodic



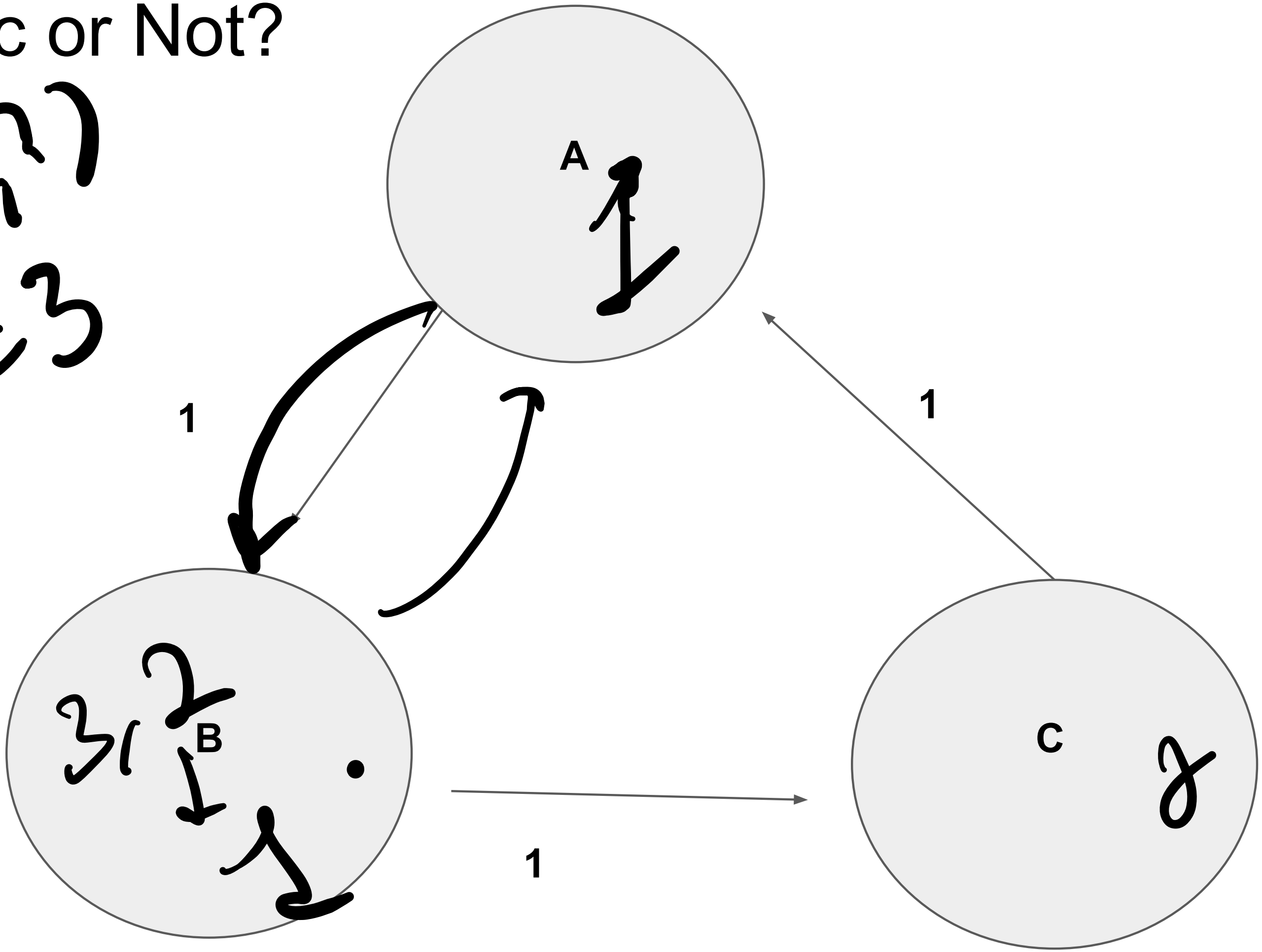
Aperiodic or Not?

aperiodic

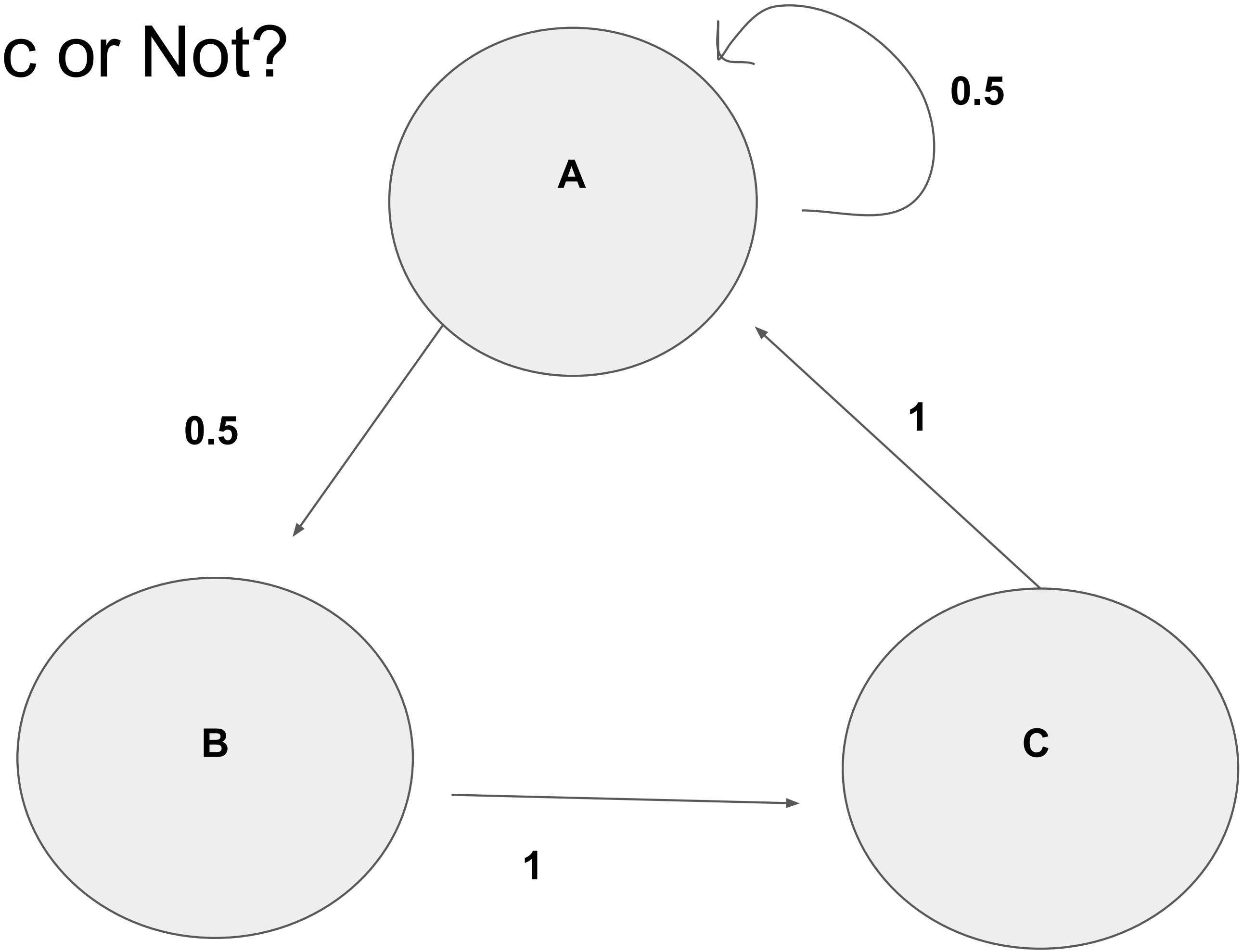


Aperiodic or Not?

gcd(3, 3, ?) = 3



Aperiodic or Not?



More Vocab!

$\pi P = \pi$

If a Markov Chain is **finite** and **irreducible**, then its invariant distribution both **exists** and **is unique**. If it is **aperiodic** as well, it **converges to the invariant distribution.**

regardless of π

$\pi P^n \rightarrow$ invariant

more than one

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving for Invariant

1. Write out transition probability matrix

State you end in

State you started at

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

2. Write out detailed balance equations – using π = πP and the fact that the starting probabilities have to add to 1.

3. Solve :)

$\pi(1)+\pi(2)+\ldots\pi(n)=1$

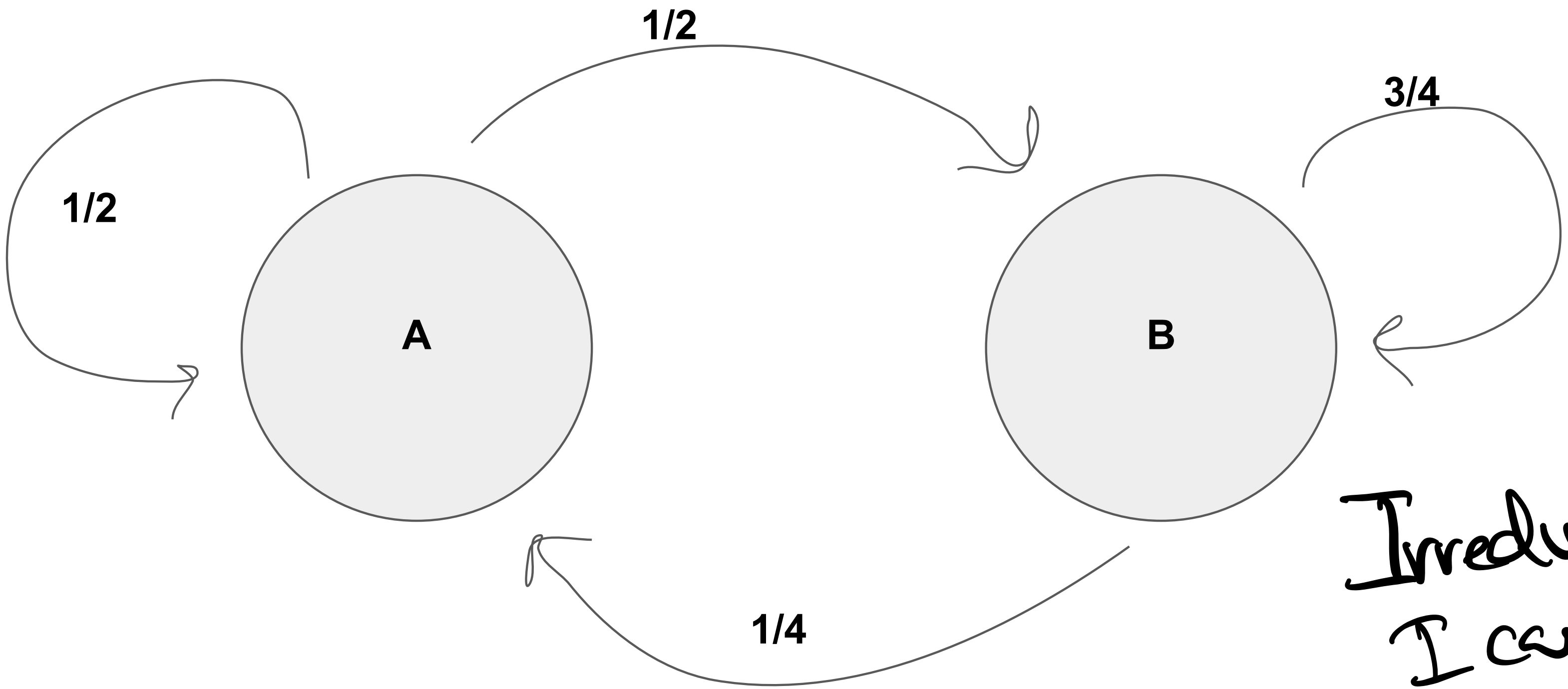
$\pi P = \pi$

do math

$$[\pi(1) \quad \pi(2) \quad \ldots \quad \pi(n)] \begin{bmatrix} P(1,1) & \cdots & \\ P(2,1) & & \\ \vdots & & \end{bmatrix}$$

$\pi(1)P(1,1)+\pi(2)P(2,1)+\ldots = \pi(1)$

Solving for Invariant



Irreducible:
I can get from A to B & vice-versa

Solving for Invariant

$$\begin{array}{c|cc} & A & B \\ A & \frac{1}{2} & \frac{1}{2} \\ B & \frac{1}{4} & \frac{3}{4} \end{array}$$

Aperiodic: self-loop

probability of starting in state A

$$\pi P = \pi$$

$$\begin{bmatrix} \pi(A) & \pi(B) \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \pi(A) & \pi(B) \end{bmatrix}$$

3/4

$$\left[\tfrac{1}{2}\pi(A)+\tfrac{1}{4}\pi(B) \quad \tfrac{1}{2}\pi(A)+\tfrac{3}{4}\pi(B)\right] = \left[\pi(A) \quad \pi(B)\right]$$

$$\tfrac{1}{2}\pi(A)+\tfrac{1}{4}\pi(B) = \pi(A) \longrightarrow \tfrac{1}{2}\pi(A) = \tfrac{1}{4}\pi(B)$$

$$\pi(A) = \tfrac{1}{2}\pi(B)$$

$$\tfrac{1}{2}\pi(A)+\tfrac{3}{4}\pi(B) = \pi(B)$$

$$\pi(A)+\underline{\pi(B)} = 1$$

$$\tfrac{1}{2}\pi(B)+\pi(B) = 1$$

$$\tfrac{3}{2}\pi(B) = 1$$

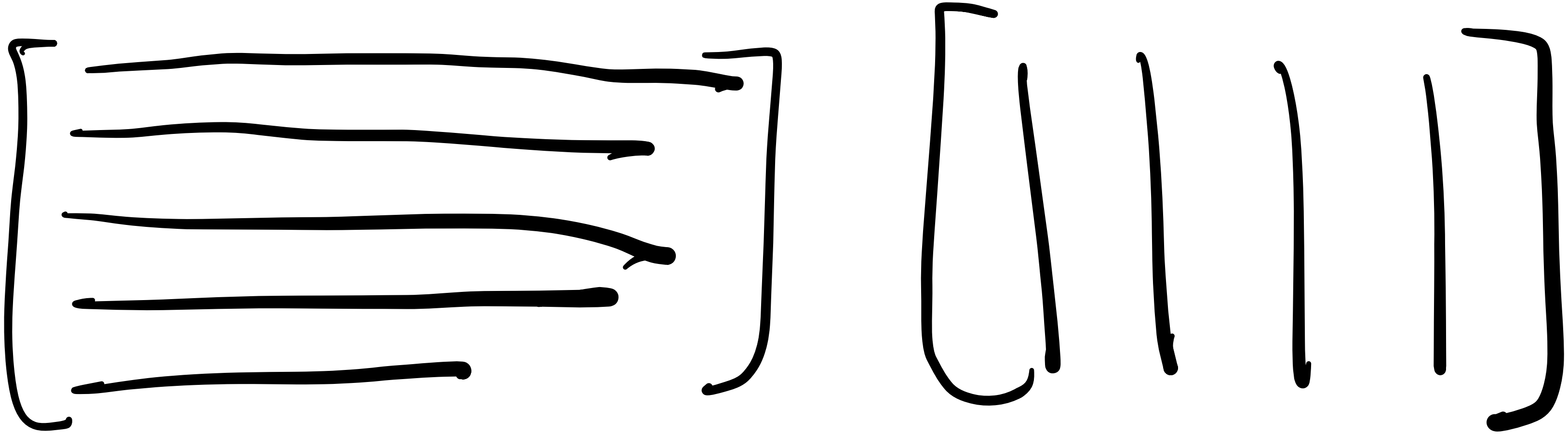
$$\pi(B) = \tfrac{2}{3}$$

$$\pi(A) = \tfrac{1}{3}$$

$\left[\tfrac{1}{3} \quad \tfrac{2}{3}\right] \rightarrow$ invariant dist.

Special Case: Doubly Stochastic Matrices

A matrix is **doubly stochastic** if all the rows and all the columns sum to 1.



invariant is unique

Doubly Stochastic Invariant

and the prob. matrix is doublystochastic

Suppose you have an **irreducible, aperiodic** finite Markov Chain. Prove that the uniform distribution is the invariant distribution.

$$\begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \begin{bmatrix} P(1,1) & \cdots & P(1,n) \\ P(2,1) & & \\ \vdots & & \\ P(n,1) & \cdots & P(n,n) \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}$$

if this

$$\frac{1}{n}P(1,1)+\frac{1}{n}P(2,1)+\dots\frac{1}{n}P(n,1)=\frac{1}{n}$$ is $\frac{1}{n}$

$$\frac{1}{n}(\underbrace{P(1,1)+P(2,1)+\dots P(n,1)})$$

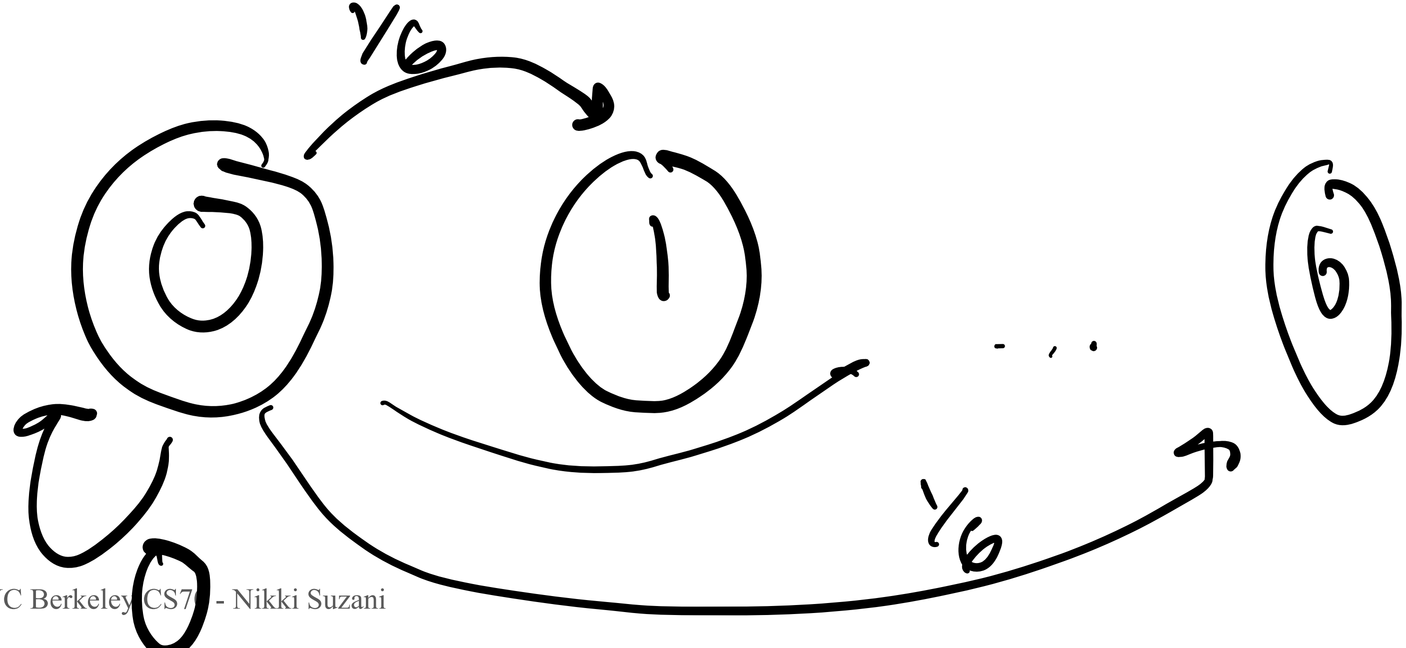
$$\frac{1}{n}(1)=\frac{1}{n}$$ ✓

if all columns sum to 1, invariant is the uniform dist.

Doubly Stochastic Example! data140 :)

You roll a dice **n** times for some large **n**. What's the probability that the sum of all the dice rolls is a multiple of 7?

X: sum of my dice rolls (mod 7)



Doubly Stochastic Example!

$$
\begin{array}{c|ccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1 & 1/6 & 0 & & & & & \\
2 & \vdots & & 0 & & & & \\
3 & \vdots & & & 0 & & & \\
4 & \vdots & & & & 0 & & \\
5 & & & & & & 0 & \\
6 & 1/6 & & & & & & 0
\end{array}
$$

all columns sum to one

Probability my sum is $0 \pmod 7$ $= 1/7$

$$\begin{bmatrix} 1/7 & \cdots & 1/7 \end{bmatrix}$$

Recap

Learned about Markov Chains:

- Represent states and transitions where the next state **only** depends on the previous state
- Learned about **invariant** (or stationary) distributions, and how irreducible/aperiodic relates to them
- Learned how to solve for stationary distributions in general, and especially in **doubly stochastic** matrices