Lecture 6A: Markov Chains

UC Berkeley CS70
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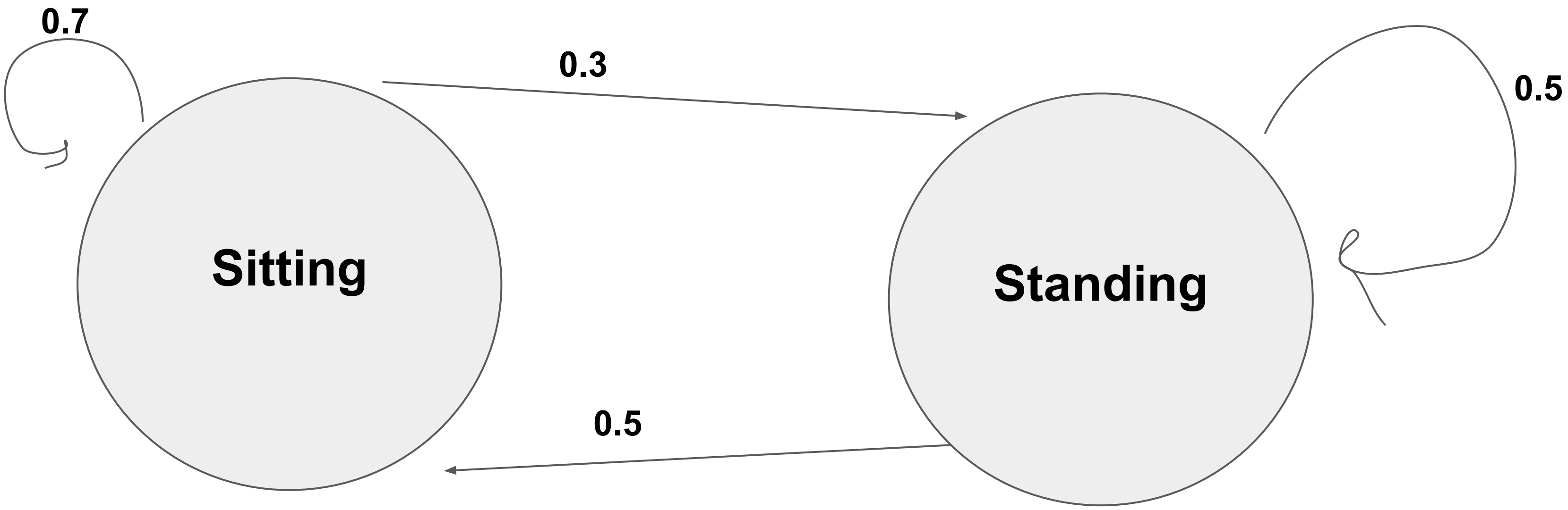
Finite Markov Chains

Markov Chains help us to visualize **transitions** between different **states.**



Finite Markov Chains (Formally)

Markov Chains are defined by two key components:

- State Space – set of all possible **states** that your experiment can exist in (all possible values of the random variable in the Markov Chain). i.e. {sitting, standing}.

Finite Markov Chains (Formally)

Markov Chains are defined by two key components:

- Transition Probability Matrix – the probability of moving from any **one state** to any **other state**

Key Information!

The **most** important thing about Markov chains, is that the chain is **amnesic (memoryless)** meaning each step should **only** depend on the previous state.

Can the following be modeled as a Markov Chain?

You have a coin that you flip continuously until you get **two consecutive** heads.

Modeled as a Markov Chain?

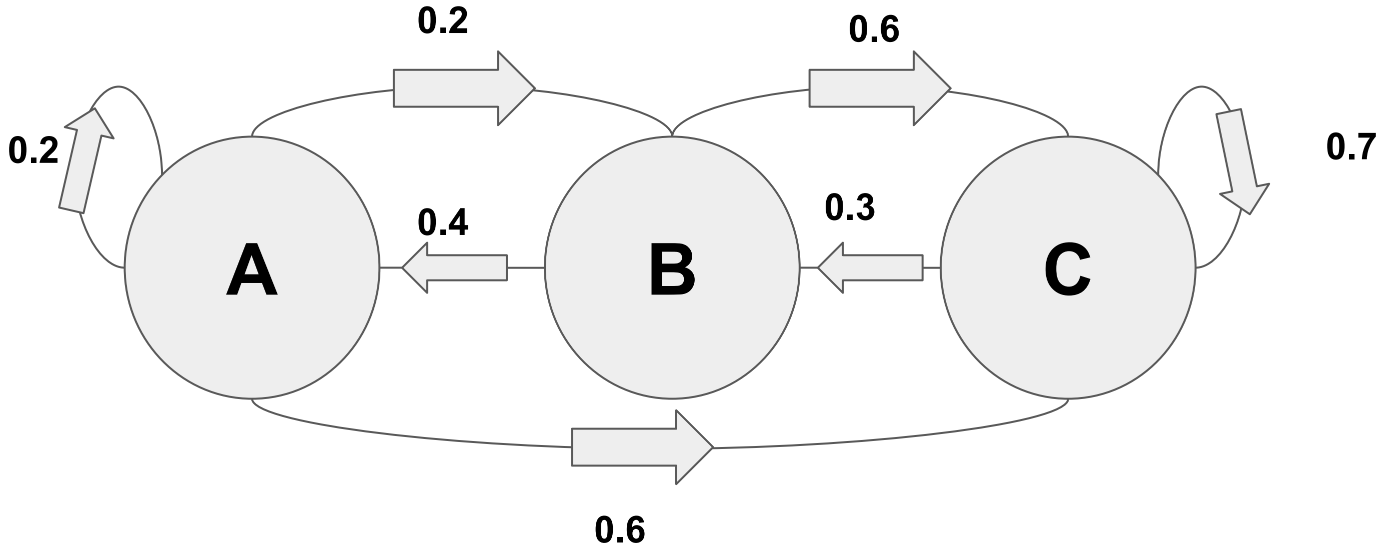
You're drawing **without replacement** from a bag with a purple, red, and green ball. Model the outcomes of your draws.

More Vocab!

Often for Markov Chain questions you're given an **initial distribution**. This tells you the probability of being in **each state** when you start. Each probability should be >= 0, and they should sum to 1.

Using the initial distribution

For some Markov Chain, let π = [0.3 0.4 0.2 0.1] for states A, B, and C respectively.



Using the initial distribution

Let $\pi = [0.3\ 0.4\ 0.2\ 0.1]$. After **one step**, what is the probability of being in **state A?**

Generalize!

Let’s look at the new matrix that we make. Now, what’s the probability of being in State A after **two steps**?

Generalize

How do we find the probability of being in a certain state after **n steps**?

More Vocab!

An **invariant (or stationary)** distribution π is a starting distribution such that:

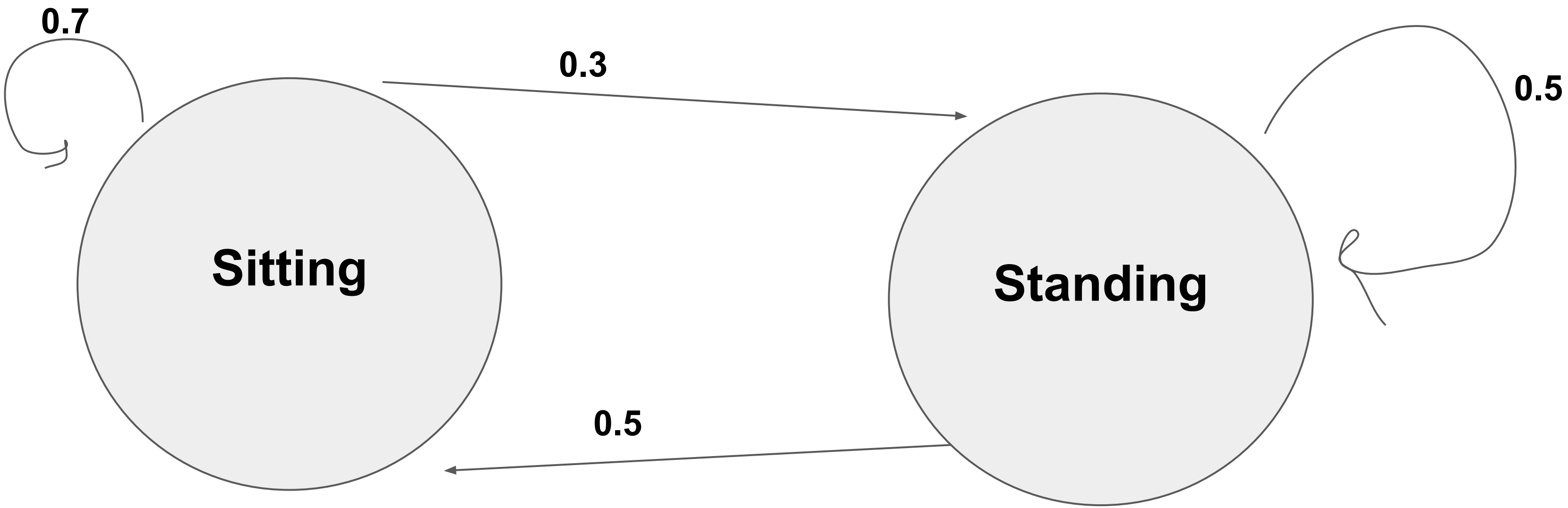
- $\pi = \pi P$

More Vocab!

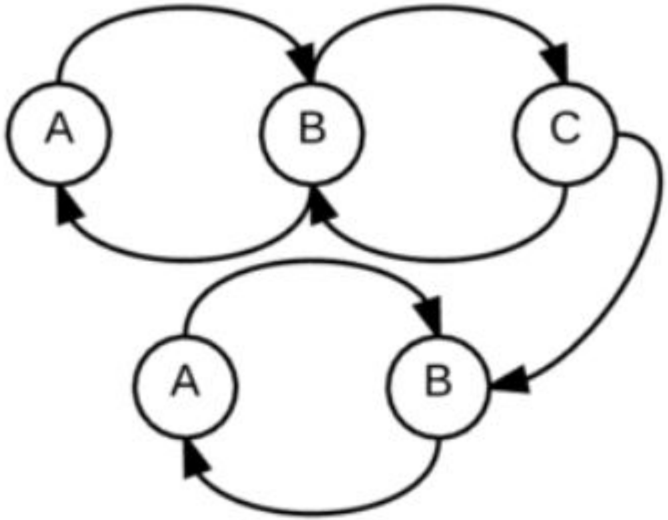
A Markov chain is said to be **irreducible** if you can go from any **starting space** to any other **ending space**.

- Specifically, there is a path of nonzero (positive) probability that goes from state **i** to state **j** for all i & j.

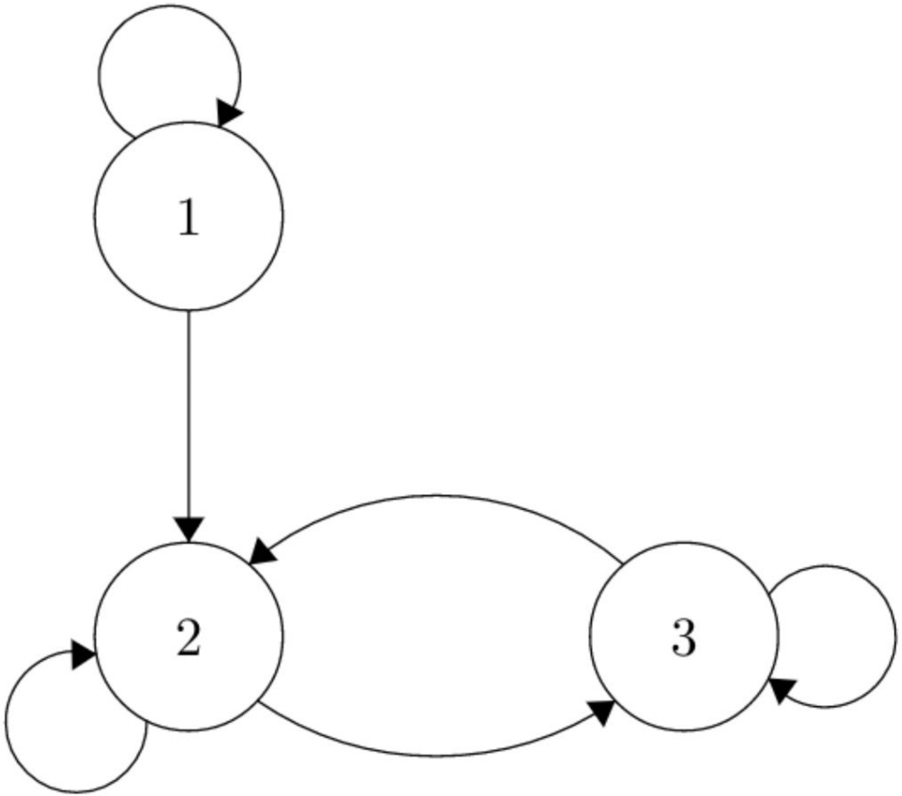
Irreducible or Not?



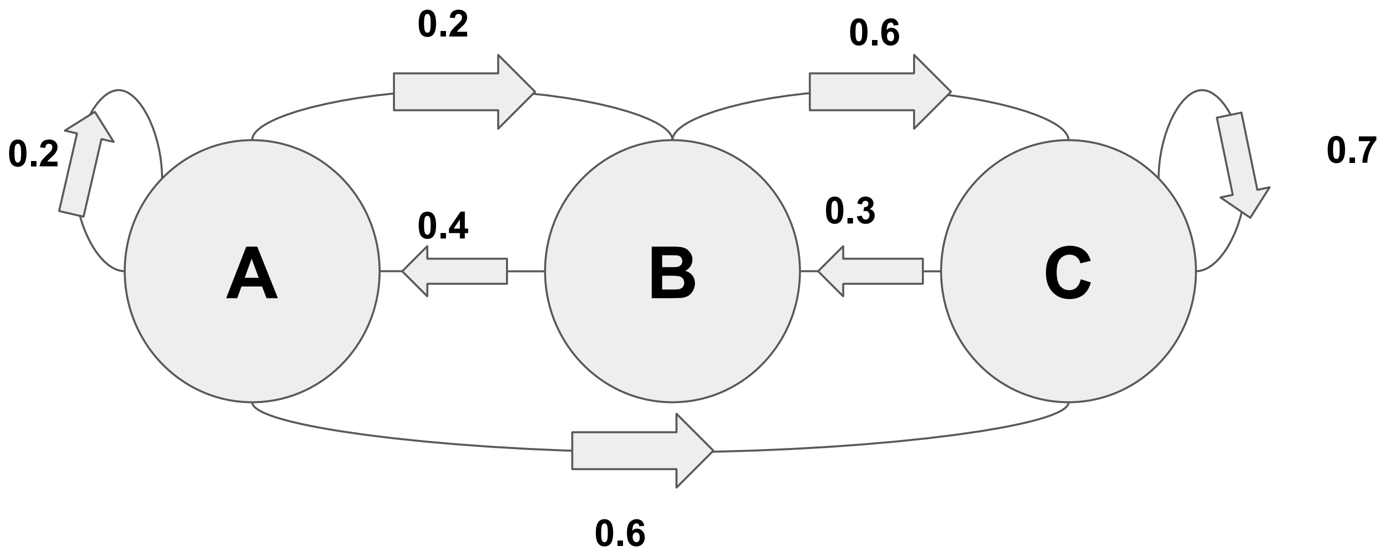
Irreducible or Not?



Irreducible or Not?



Irreducible or Not?



More Vocab!

A **state** i has **period** d if the Markov Chain can only come back to **i** in multiples of **d.**

- A Markov Chain is said to be **aperiodic** if the period of the chain is 1 – meaning at any step, you have nonzero probability of being in any state.

Aperiodic (Formally)

Theorem 22.4. *Consider an irreducible Markov chain on $\mathscr{X}$ with transition probability matrix P. Define*

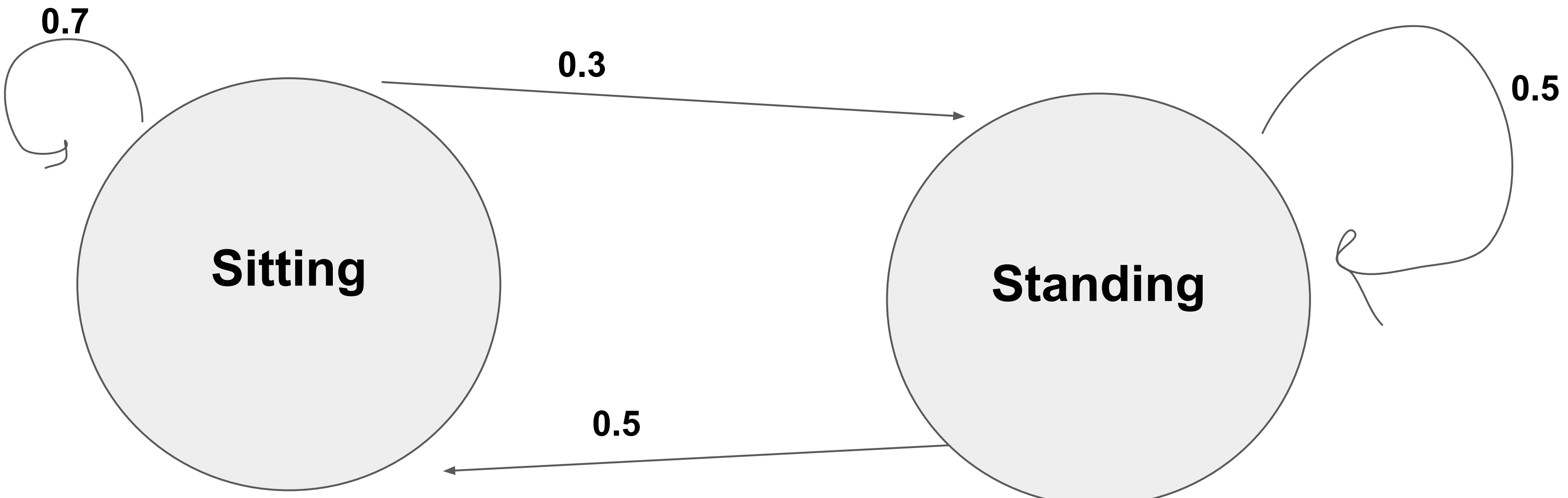
$$d(i) := g.c.d\{n > 0 \mid P^n(i,i) = Pr[X_n = i | X_0 = i] > 0\}, i \in \mathscr{X}. \quad (16)$$

(a) Then, $d(i)$ has the same value for all $i \in \mathscr{X}$. If that value is 1, *the Markov chain is said to be* aperiodic. *Otherwise, it is said to be* periodic with period d.

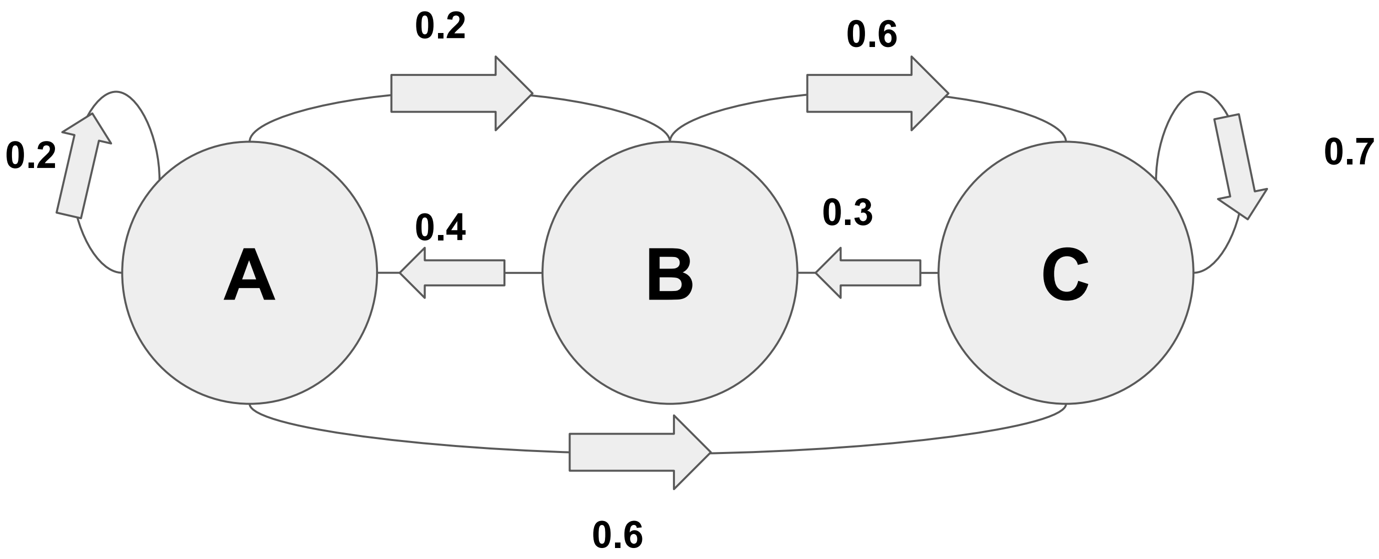
Aperiodic Quick Tip!

If any state in an irreducible Markov Chain has a **self-loop with nonzero probability**, then it is **aperiodic**.

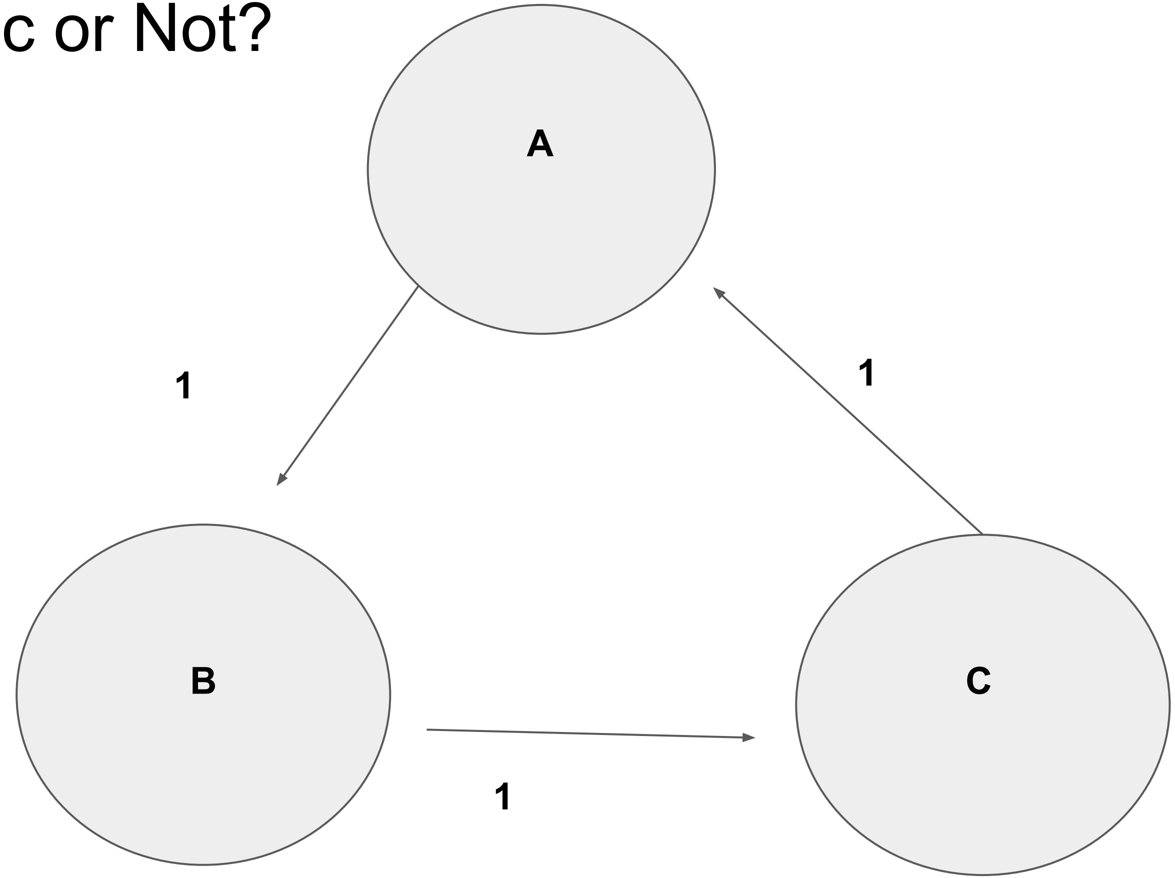
Aperiodic or Not?



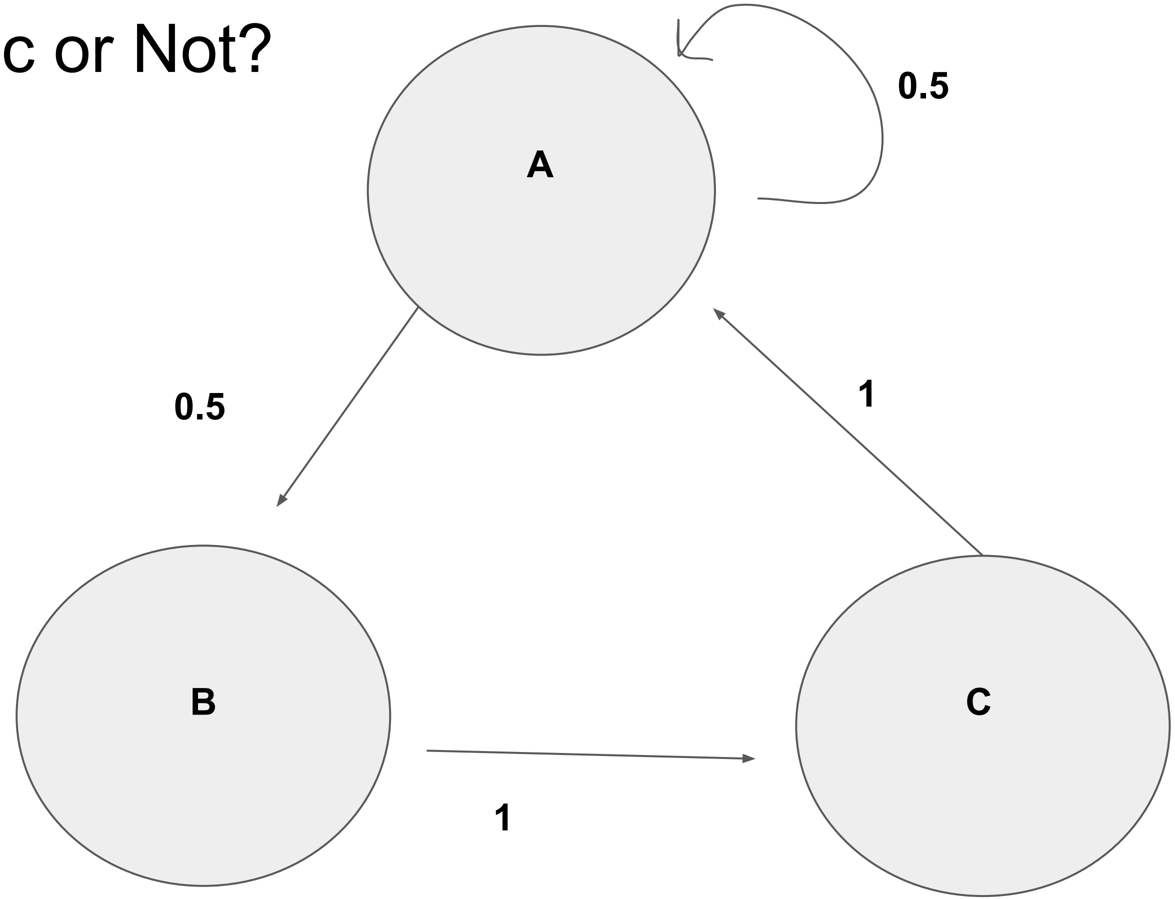
Aperiodic or Not?



Aperiodic or Not?



Aperiodic or Not?



More Vocab!

If a Markov Chain is **finite** and **irreducible**, then its invariant distribution both **exists** and **is unique**. If it is **aperiodic** as well, it **converges to the invariant distribution.**

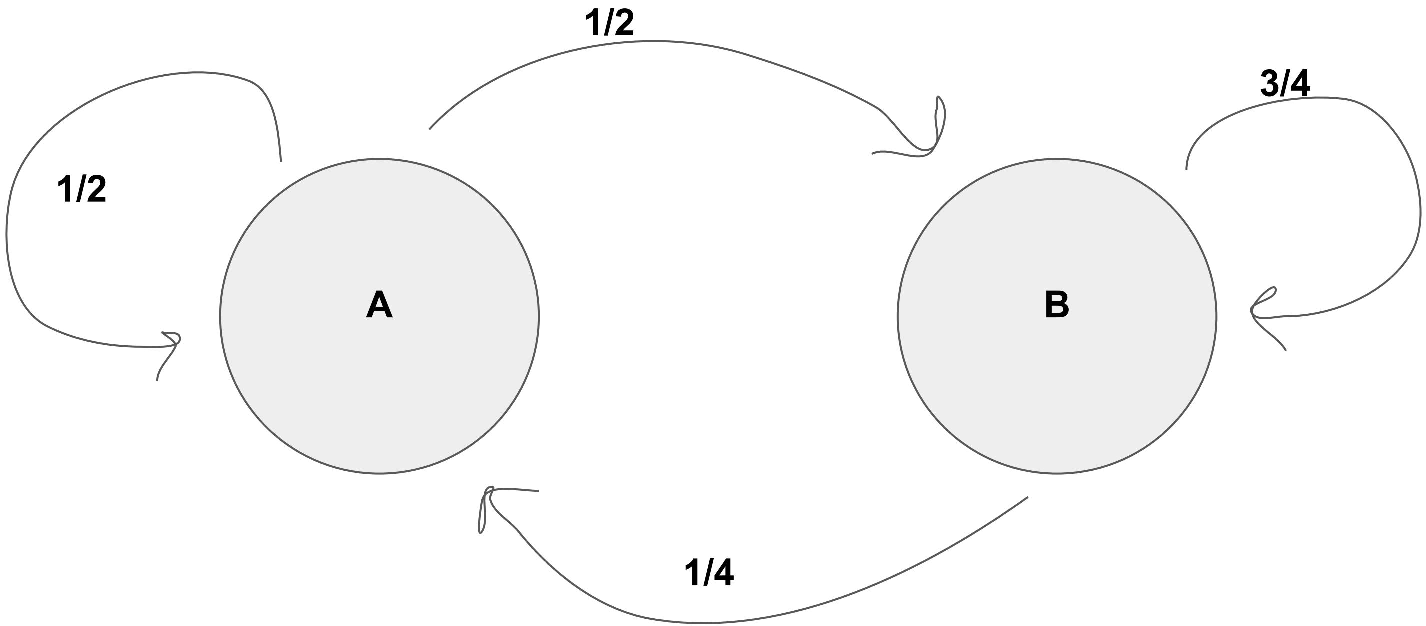
Solving for Invariant

1. Write out transition probability matrix

2. Write out detailed balance equations – using $\pi = \pi P$ and the fact that the starting probabilities have to add to 1.

3. Solve :)

Solving for Invariant



Solving for Invariant

Special Case: Doubly Stochastic Matrices

A matrix is **doubly stochastic** if all the rows and all the columns sum to 1.

Doubly Stochastic Invariant

Suppose you have an **irreducible, aperiodic** finite Markov Chain. Prove that the uniform distribution is the invariant distribution.

Doubly Stochastic Example!

You roll a dice **n** times for some large **n**. What's the probability that the sum of all the dice rolls is a multiple of 7?

Doubly Stochastic Example!

Recap

Learned about Markov Chains:

- Represent states and transitions where the next state **only** depends on the previous state
- Learned about **invariant** (or stationary) distributions, and how irreducible/aperiodic relates to them
- Learned how to solve for stationary distributions in general, and especially in **doubly stochastic** matrices