CS 70 Su23: Lecture 24 (6B)

Markov Chains II: hitting time, probability of state A before state B

A quick recap

Markov chain (it’s a directed graph), defined by:

- States (nodes)
- Transition probabilities (edges)
  - self-edges are allowed
  - it’s probability, so all the outgoing edges have to add to 1
    - do all incoming edges have to add to 1?[1]
- Initial probabilities (how likely you are to start at each state)

Markov chains are **amnesic**: the transition probabilities only depend on the current state

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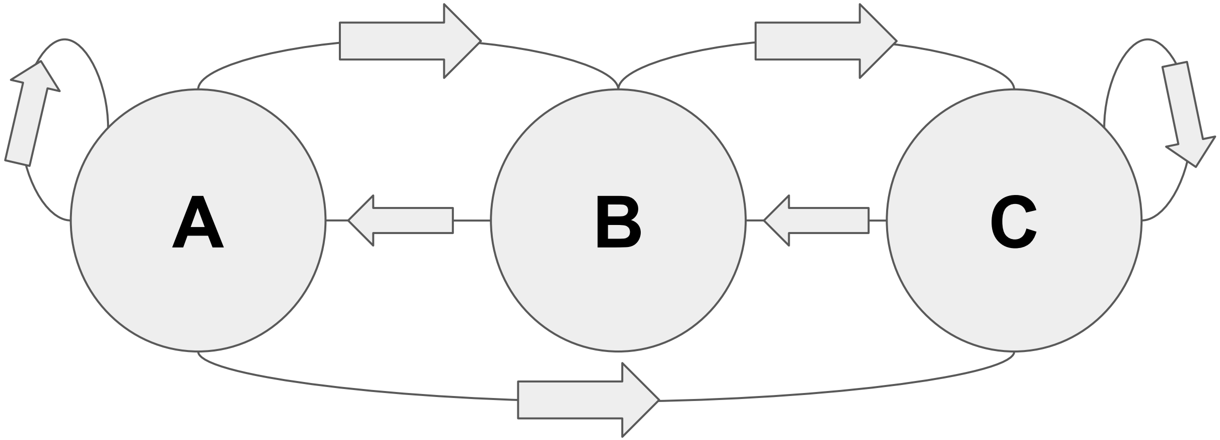
Hitting time

Given a starting state in a Markov Chain, what is the expected number of steps (transitions) until we reach some other state?



Suppose that:

- Pr[A→B] = 1
- Pr[B→C] = 1

If we start at **A**, how many steps, on average, will it take to get to C?

- 2 steps

First-step equations

Random variables are our friend. Let's define:

- T_{AB} = the number of steps it takes to go from state A to state B

If you only care about one of the states, you can simplify this (for example):

- T_A = the number of steps it takes to go from A to C
    - this is useful shorthand if your markov chain has a clear "end state"

We are interested in computing $E[T_A]$, for some state A (to reach C).

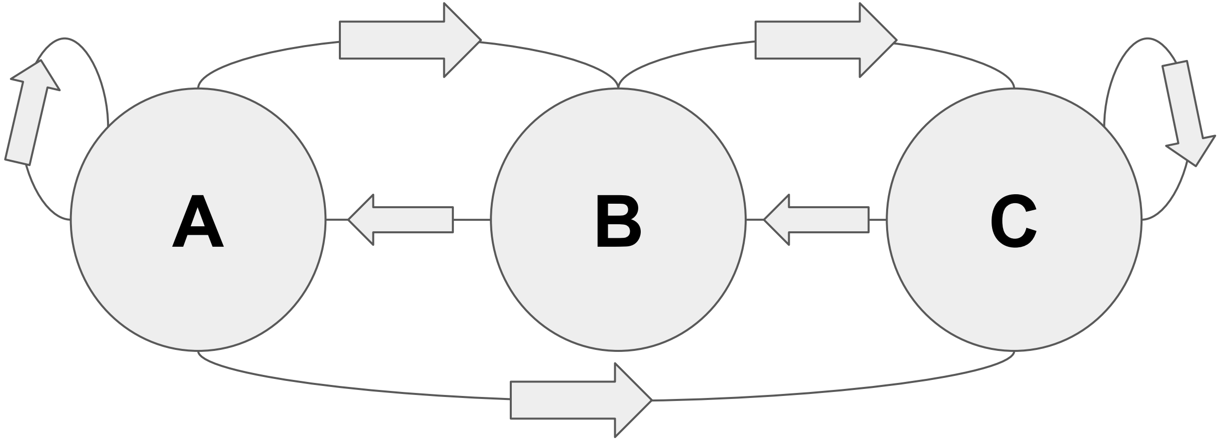
You will also see this as a function (the notes uses β):

- T(A) = the *expected* number of steps to reach state C from state A
    - Note: this is an expectation! Be careful when reading to double check what the variable represents

First-step equations

What is T(C)?

- $E[T_{CC}] = 0$ (you're already there!)



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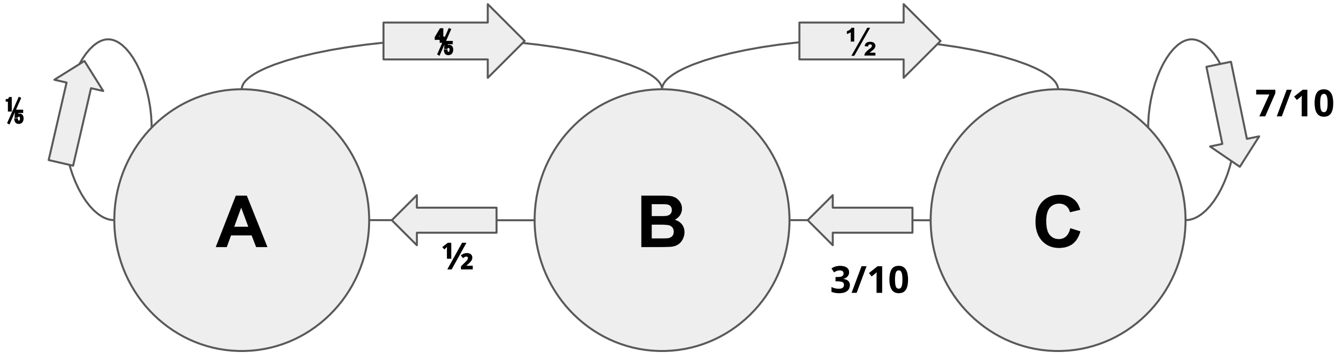
Example 1: first-step equations (with numbers)

What is T(A)?

- $T(A) = 1 + \frac{1}{5}T(A) + \frac{4}{5}T(B)$

What is T(B)?

- $T(B) = 1 + \frac{1}{2}T(A) + \frac{1}{2}T(C)$



Example 1: first-step equations (with numbers)

$T(A) = 1 + \frac{1}{5}T(A) + \frac{4}{5}T(B)$

$T(B) = 1 + \frac{1}{2}T(A) + \frac{1}{2}T(C)$

$= 1 + \frac{1}{2}T(A)$

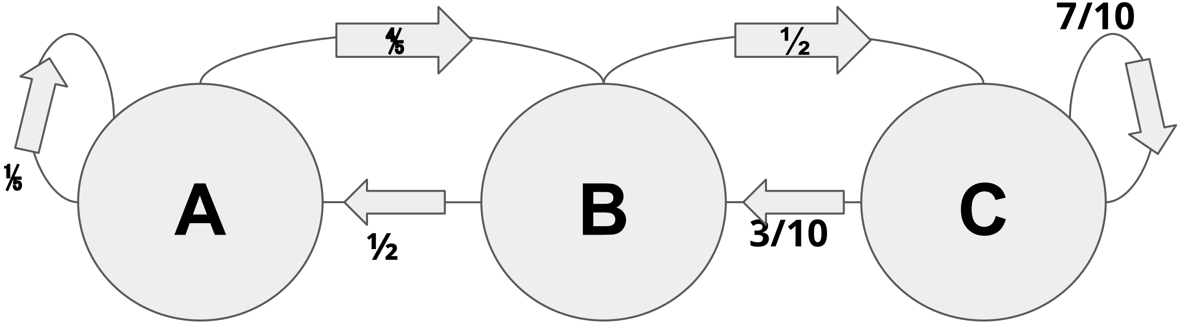
$T(A) = 1 + \frac{1}{5}T(A) + \frac{4}{5}(1 + \frac{1}{2}T(A))$

$= 1 + \frac{1}{5}T(A) + \frac{4}{5} + \frac{2}{5}T(A)$

$= 9/5 + \frac{3}{5}T(A)$

$\frac{2}{5}T(A) = 9/5$

$T(A) = 9/2$



First-step equations

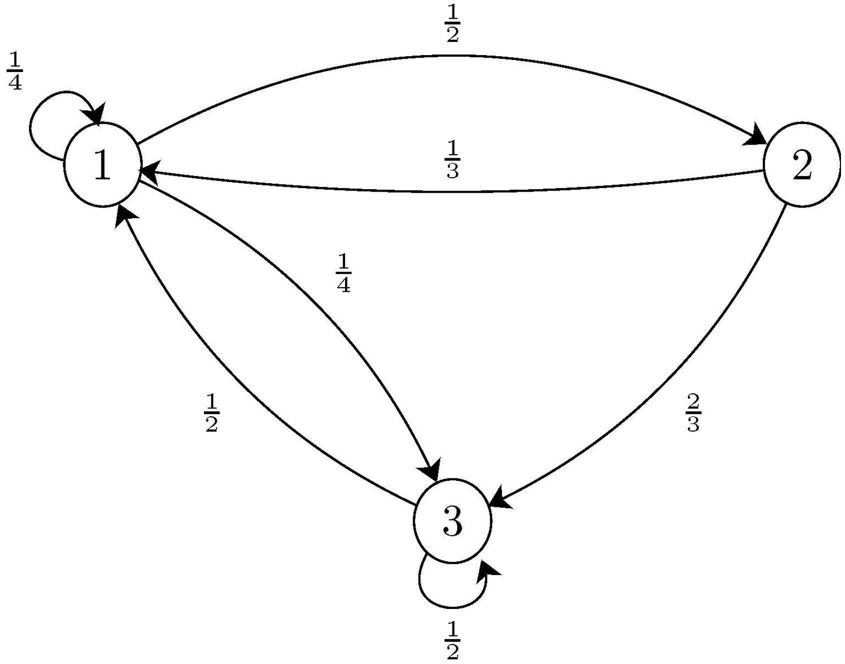
In general, the expected number of steps to state Y from state X is:

- $T_Y(X) = 1 + \sum_Z \Pr[X \to Z] \cdot T_Y(Z)$
- $T_Y(Y) = 0$

In other words, we have to go somewhere, and with some probability, we end up at a different state, which takes some expected number of steps to get to state Y (if you start when you want to go, it takes 0 steps to get there).

(again, if Y is implied, you can omit it)

Example 2: more numbers



We want to reach state 1. What is $T_1(2)$?

$T(2) = 1 + \frac{1}{3}T(1) + \frac{2}{3}T(3)$

$= 1 + \frac{2}{3}T(3)$

$T(3) = 1 + \frac{1}{2}T(1) + \frac{1}{2}T(3)$

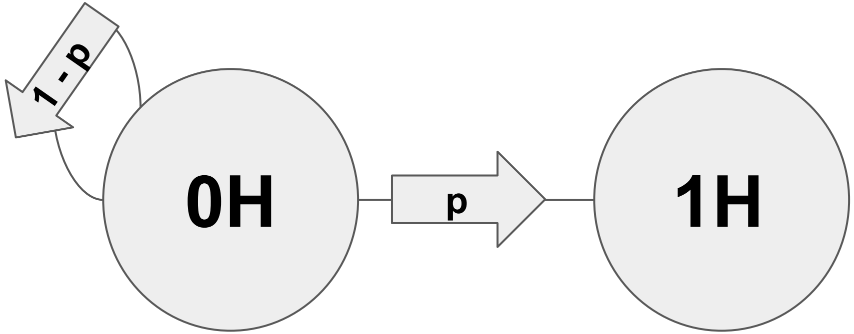
$= 1 + \frac{1}{2}T(3)$

$T(3) = 2$

$T(2) = 7/3$

Example 3: Geometric... again

How can we represent a geometric distribution as a Markov Chain?



$$T(0) = 1 + pT(1) + (1 - p)T(0)$$

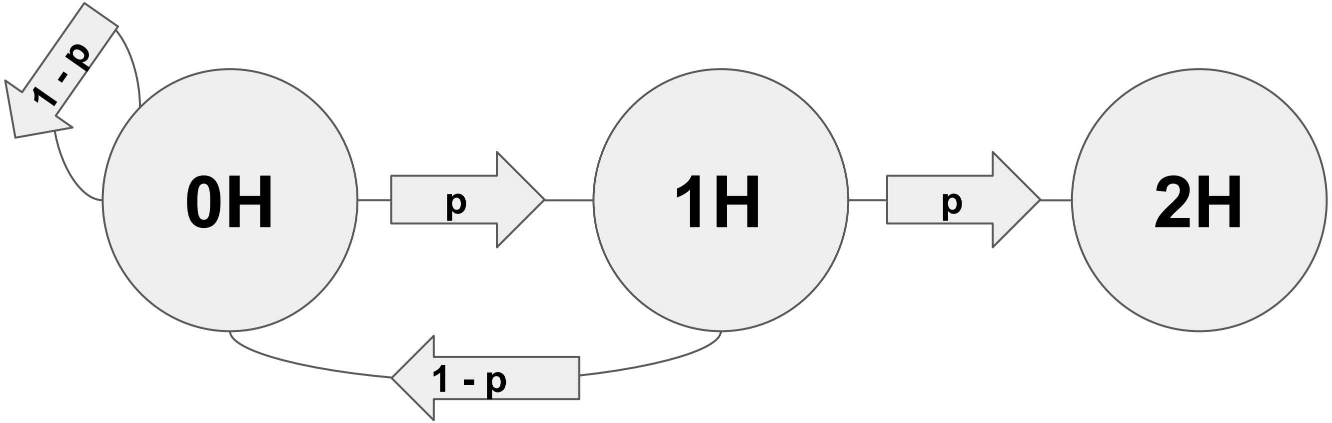
$$= 1 + T(0) - pT(0)$$

$$pT(0) = 1$$

$$T(0) = 1/p$$

Example 4: Geometric… again (again)

Now what if we want to flip until we get 2 heads in a row?

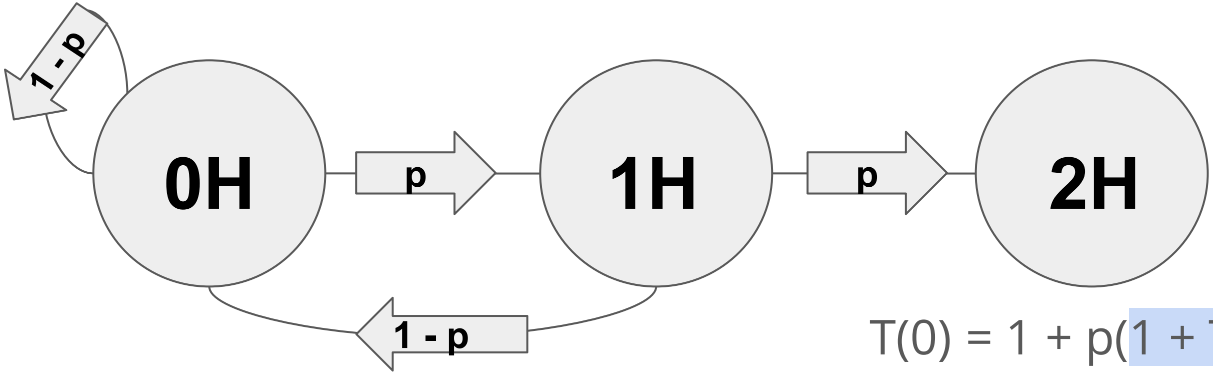


Example 4: Geometric... again (again)

$$T(0) = 1 + pT(1) + (1 - p)T(0)$$

$$T(1) = 1 + pT(2) + (1 - p)T(0)$$

$$= 1 + T(0) - pT(0)$$



$$T(0) = 1 + p(1 + T(0) - pT(0)) + (1 - p)T(0)$$

$$T(0) = 1 + p + pT(0) - p^2T(0) + T(0) - pT(0)$$

$$T(0) = (1 + p)/p^2$$

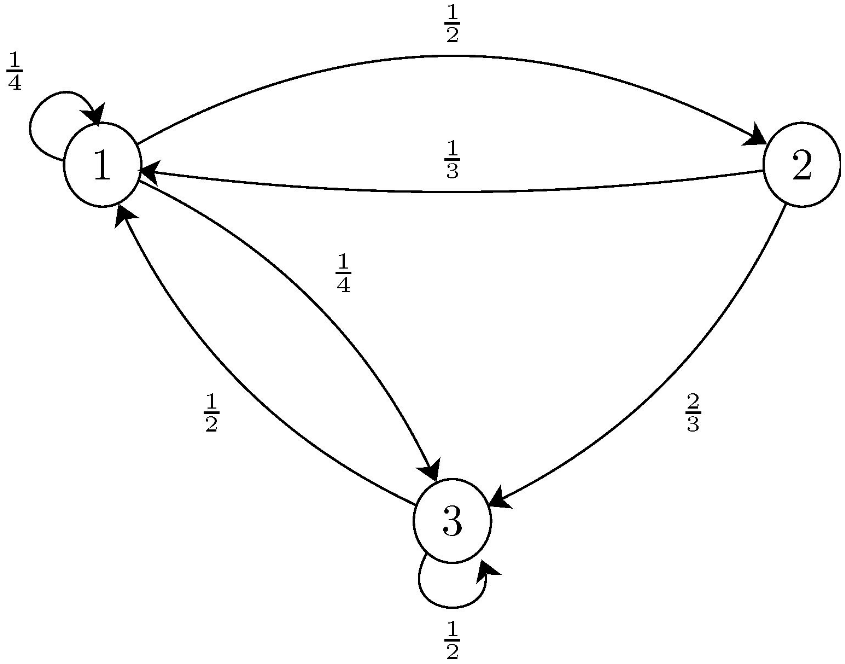
Return time

We start at state X. What is the expected number of steps until we *return* to state X?

- $R_X = 1 + \sum_Z \Pr[X \to Z] \cdot T_X(Z)$

In other words, we have to go somewhere, and with some probability, we end up at a different state, which takes some expected number of steps to get to state 1.

Example 5: the numbers from before



We start in state 1. What is R_1?

$$
\begin{aligned}
R_1 &= 1 + \tfrac{1}{4} T(1) + \tfrac{1}{2} T(2) + \tfrac{1}{4} T(3) \\
&= 1 + 0 + \tfrac{1}{2} \cdot 7/3 + \tfrac{1}{4} \cdot 2 \\
&= 6/6 + 7/6 + 3/6 \\
&= 16/6 \\
&= 8/3
\end{aligned}
$$

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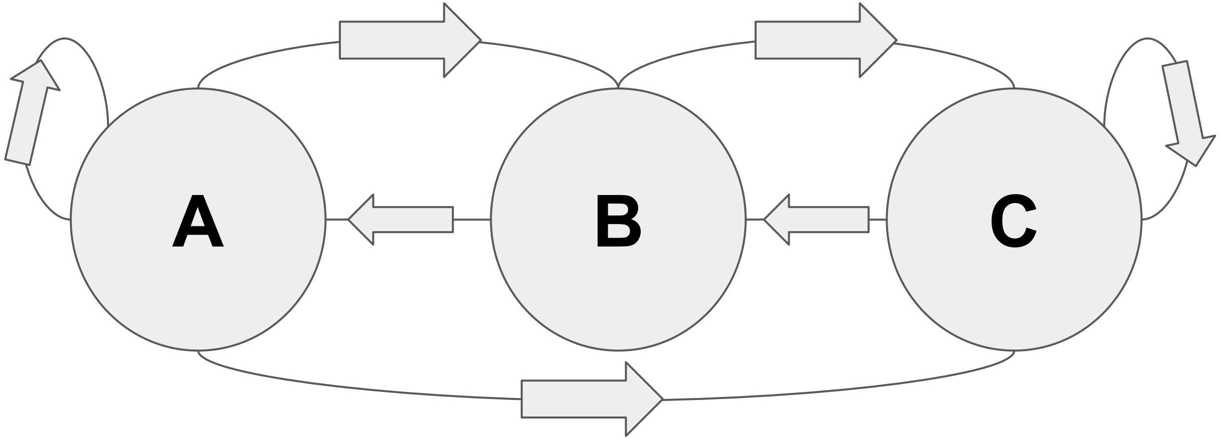
Probability of state A before state B

Given a starting state, what is the probability we reach some state before some other state?



Suppose that:

- Pr[A→B] = 1
- Pr[B→C] = 1

If we start at **A**, what is the probability of reaching C before B?

- 0

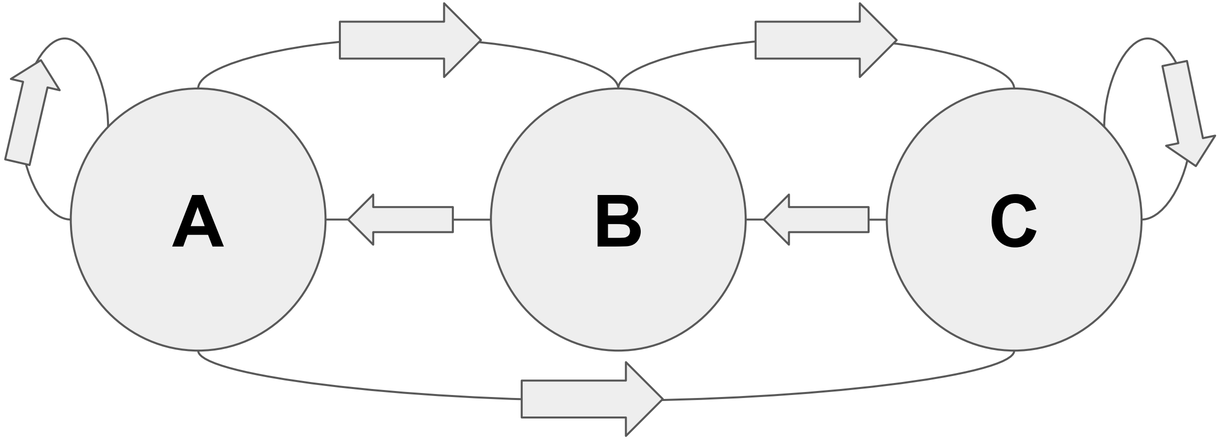
Probability of state A before state B

Let P(X) = the probability that you reach state A before state B from state X (for some specified states A and B).

- This is, again, a function (the notes uses α)
- If you want to be very precise, you can use subscripts for clarity
  - Example: $P_{CB}(X)$ = probability if reaching C before B, starting from X

Example 6: probability of C before B

Returning to our example:



Let's say we are interested in whether we visit C before B.

What is P(C)?

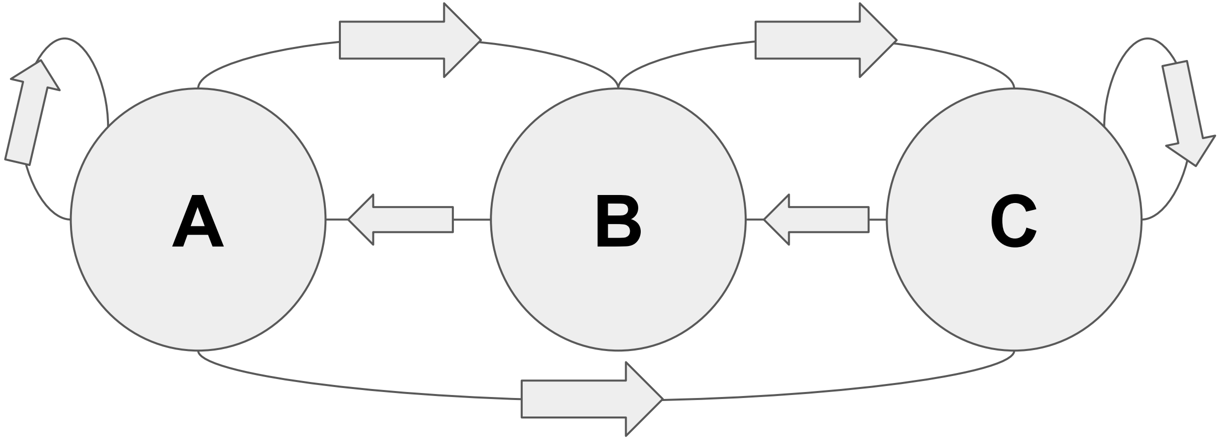
- 1

What is P(B)?

- 0

Example 6: probability of C before B

Returning to our example:



What is P(A)?

$$P(A) = \Pr[A \to A] \cdot P(A) + \Pr[A \to B] \cdot P(B) + \Pr[A \to C] \cdot P(C)$$

$$= \Pr[A \to A] \cdot P(A) + \Pr[A \to C] \cdot P(C)$$

Probability of state A before state B

In general, the probability of reaching state A before state B, starting from state X, is:

- $P_{AB}(X) = \sum_Y \Pr[X \rightarrow Y] \cdot P_{AB}(Y)$
- $P_{AB}(A) = 1$
- $P_{AB}(B) = 0$

(again, if A and B are implied, you can omit them)

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Example 7: gamble responsibly

You are playing a game where you flip a coin with probability p. If it lands heads, you win \$1. Otherwise, you lose \$1. The game ends when you have \$0. What is the probability that you earn \$N before you run out of money?

Let's write out the equations:

- $P(0) = 0$
- $P(N) = 1$
- $P(n) = p \cdot P(n + 1) + (1 - p) \cdot P(n - 1)$, for $0 < n < N$

With the magic of algebra, we have $P(n) = (1 - ((1 - p)/p)^n) / (1 - ((1 - p)/p)^N)$