Lecture 6C: Continuous Probability I

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Continuous Probability

Continuous probability is different from discrete probability. How so? Let's try to visualize it.

I'm going to write down a number from the following set {1, 2, 3, 4}, uniformly at random. Try to guess the number I wrote down.

Continuous Probability

Now, I’m going to write down a number from [1, 4], uniformly at random. Try to guess the number I wrote down. This should be much harder.

Continuous Probability

In our continuous probability example, it makes more sense to think about probabilities in terms of an interval, rather than an exact point.

Let’s try this again, but this time, try to predict if my number is in the following ranges [1, 2), [2, 3), [3, 4].

Continuous Probability

In our continuous probability example, it makes more sense to think about probabilities in terms of an interval, rather than an exact point.

Let's try this again, but this time, try to predict if my number is in the following ranges [1, 2), [2, 3), [3, 4].

So what's the point of this? Sometimes we run into situations where we want to use probabilities in an interval or <u>continuum</u>. We call these cases continuous probability.

Probability Mass Function

Def: A probability mass function (p.m.f.) is the function $p: R \to [0, 1]$ defined by

$$p_X(i) = P[X = i]$$

Probability Density Functions

Def: A probability density function is a function f: R → R which describes

$$\mathbb{P}[a \leq X \leq b] = \int_a^b f(x)dx \qquad \textit{for all } a < b.$$

Properties of Probability Density Functions

Two properties need to be satisfied by PDFs:

1. f is non-negative: $f(x) \geq 0$ for all $x \in R$
2. The total integral of f is equal to 1: $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

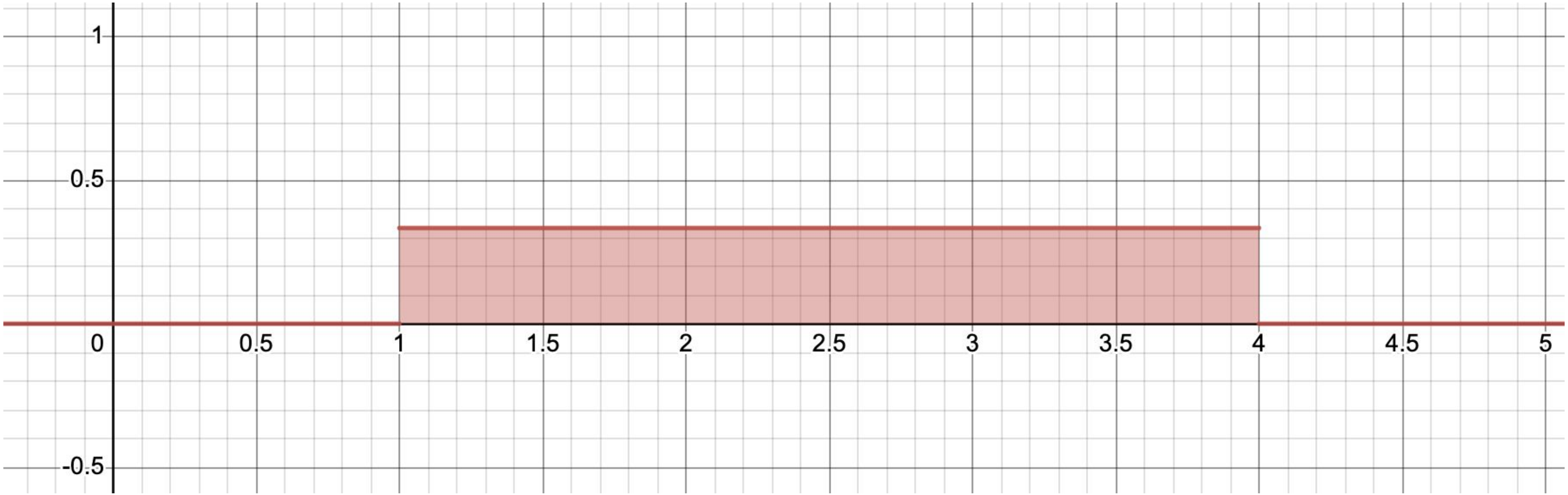
These properties should make sense, keeping in mind what a PDF represents.

1. Probabilities must be non-negative, so a PDF must also be non-negative.
2. $P[-\infty \leq X \leq \infty] = 1$ (X is real valued), so the total integral must equal 1

Example PDF

Let's describe the PDF of a random variable X ~ uniform distribution on the interval [0, L].

Example PDF



An interpretation of “density”

A useful interpretation of the PDF is when we consider a small interval [x, x+dx].

$$\mathbb{P}[x \leq X \leq x + dx] = \int_x^{x+dx} f(z)dz \approx f(x)dx.$$

f(x)*dx is the area of a rectangle, with width dx and height f(x). In this sense, f(x) is the “probability per unit length” around x.

Cumulative Distribution Function

The cumulative distribution function (CDF) is another useful way to look at the distribution of a random variable X.

Def: The cumulative distribution function (CDF) is the function F defined as

$$F(x) = \mathbb{P}[X \leq x] = \int_{-\infty}^{x} f(z)\,dz.$$

Connection between CDF and PDF

The CDF represents the cumulative “area under the curve” of the PDF, so knowing the PDF can tell us what the CDF is. Can we go the other direction? Yes!

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$$f(x) = \frac{dF(x)}{dx}$$

PDF

CDF

$$F(x) = \int_{-\infty}^{x} f(z)\,dz$$

Properties of the CDF

For any CDF F(x), the following must be true

1. F is a monotonically increasing (non-decreasing) function
2. $\lim_{x \to \infty} F(x) = 1$, $\lim_{x \to -\infty} F(x) = 0$
3. $0 \leq F(x) \leq 1$

Example (finding CDF of previous PDF)

Let's find the CDF of a random variable X ~ uniform on the interval [a, b].

Example (finding CDF of previous PDF)

Plug in a = 1, b = 4 to get the CDF of the example we worked with before.

Expectation of Continuous

Recall the expectation for a discrete random variable: $\mathbb{E}[X] = \sum_{a \in \mathscr{A}} a \times \mathbb{P}[X = a]$

In the continuous case, instead of summing across all possible values of X, we integrate across the entire space.

Def: The expectation of a continuous r.v. X with probability density function f is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx.$$

Variance of Continuous

Def: The variance of a continuous r.v. X with probability density function f is

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)dx - \left(\int_{-\infty}^{\infty} x f(x)dx\right)^2.$$

Example Expectation/Variance

Let X be a uniform r.v. on the interval [0, l]. Let's compute E[X] and Var(X).

E[X] =

Var(X) =

Joint & Marginal Distributions

If we have more than one random variable, say X and Y, we can describe their joint distribution in a similar way to the discrete case. Remember, we'd like to express $P[a \leq X \leq b, c \leq Y \leq d]$.

Def: A joint density function for two r.v.s X and Y is a function $f: R^2 \to R$ which is given by

$$\mathbb{P}[a \leq X \leq b,\ c \leq Y \leq d] = \int_c^d \int_a^b f(x,y)\,dx\,dy \qquad \text{for all } a \leq b \text{ and } c \leq d.$$

Joint & Marginal Distributions

A joint density function satisfies the following properties:

1. f is non-negative: $f(x, y) \geq 0$ for all $x, y \in R$
2. The total integral of f is equal to 1: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Similar to the density function of a single r.v., the joint density f(x, y) tells us the "probability per unit area" around (x, y).

Joint & Marginal Distributions

Given a joint distribution $f(x, y)$ of two r.v.s X and Y, what is the marginal distribution of X and Y?

Independence of Two Continuous RVs

Def: Two continuous r.v.'s X, Y are <u>independent</u> if the events "a ≤ X ≤ b" and "c ≤ Y ≤ d" are independent for all a ≤ b and c ≤ d:

$$\mathbb{P}[a \le X \le b,\ c \le Y \le d] = \mathbb{P}[a \le X \le b] \cdot \mathbb{P}[c \le Y \le d].$$

Independence continued

Theorem 21.1.

The joint density of independent r.v.'s X and Y is the product of the marginal densities: $f(x,y) = f_X(x)\, f_Y(y) \quad for\ all\ \ x, y \in \mathbb{R}.$

Proof:

Conditional Joint Densities

For continuous random variables X and Y, we know that

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Can we reason the following?

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

Conditional Joint Densities

Proof:

Romeo & Juliet Problem

Romeo and Juliet have a date at a given time and each will arrive at the meeting place with a delay between 0 and 1 hour. Let X, Y denote the delays of R and J respectively. All pairs of delay (x, y) are equally likely. The first one to arrive will wait 15 min and leave if the other hasn’t arrived. What’s the probability that they meet?

Integrating a Joint Density

Let X and Y be continuous random variables ranging from [0, 1] and [0, 3] respectively, such that their joint distribution if $f(x, y) = cxy$. What is c?

Example of Integrating Joint Density

Let X and Y be continuous random variables ranging from [0, 1] and [0, 3] respectively, such that their joint distribution if $f(x, y) = (4/9)xy$. What are the marginal distributions of X and Y?

Uniform Continuous Distribution

We’ve seen this already, but as a refresher, a continuous r.v. X is uniform if X takes on values in some range [a, b] s.t. the PDF of X is constant in that range, and 0 otherwise. In other words:

Expectation and Variance of Uniform

Let X be a uniform continuous r.v. on the interval [a, b].

E[X] =

Var(X) =

Example Uniform

See above examples, first example is $a=1$, $b=4$, second example is $a=0$, $b=L$.

Exponential Distribution

Recall the geometric distribution, which counts the number of trials until a success. Is there a way to translate this idea of counting the “length until an event happens” to a continuous random variable?

Let’s try to model an alarm clock. This alarm clock is terrible and works in the following way: Once plugged in, the alarm will randomly ring once after some amount of time, however we know it goes off at a rate of 1 time every 10 minutes.

How would I express the amount of time it takes for this alarm to ring? Specifically, how would I express the PDF of that random variable?

Exponential Distribution

Def: For some λ > 0, a continuous random variable X is an exponential random variable with parameter λ if it has the following PDF,

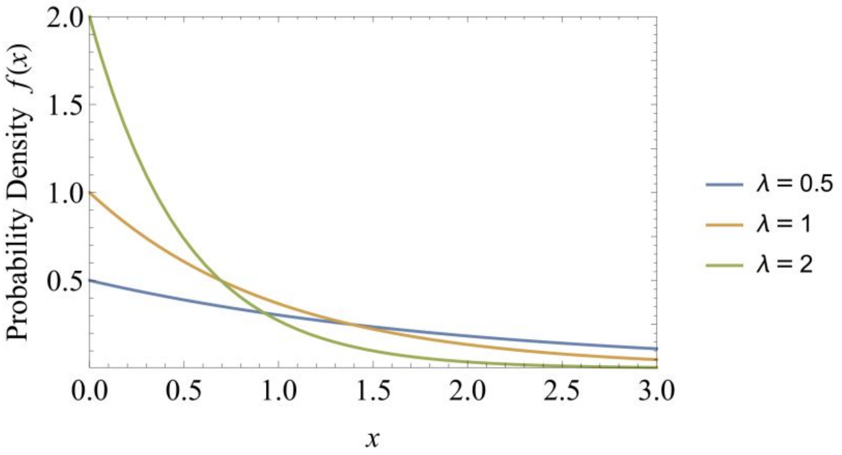
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

We can write X ~ Exp(λ) if is an exponential random variable.

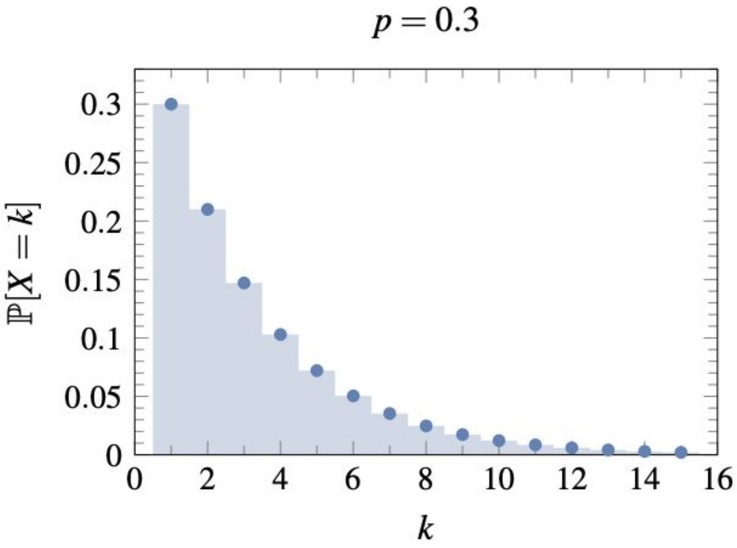
Let's check that f(x) satisfies the two PDF properties

Exponential distribution visualized

Exponential Distribution:



Geometric Distribution:



Expectation & Variance of Exponential

For a random variable X ~ Exp(λ),

$$\mathbb{E}[X] = \frac{1}{\lambda} \qquad \textit{and} \qquad \mathrm{Var}(X) = \frac{1}{\lambda^2}.$$

Proof of Expectation of an Exponential

$E[X] =$

Proof of Variance of an Exponential

Var(X) =

Example Exponential

Going back to our terrible alarm clock, we know the behavior is:

Once plugged in, the alarm will randomly ring once after some amount of time, however we know it goes off at a rate of 1 time every 10 minutes.

Let X be the amount of time it takes the alarm to sound. $X \sim Exp(1/10)$, because our rate of rings is 1/10 per minute.

How many minutes should we expect to wait before the alarm rings?

An important property to note

$$\mathbb{P}[X > t] = e^{-\lambda t}$$

Proof:

Exponential Relation to Geometric

Let's try to draw a more rigorous connection between Exponential and Geometric distributions. Take our Geometric r.v. and consider running trials after every d seconds. The probability of a success is p = λd.

Then, let Y denote the amount of time/seconds before a successful trial:

$$\mathbb{P}[Y > k\delta] = (1-p)^k = (1-\lambda\delta)^k,$$

To translate our trials to a continuum, consider taking the limit of d → 0. Then, for any time t,

$$\mathbb{P}[Y > t] = \mathbb{P}[Y > (\tfrac{t}{\delta})\delta] = (1-\lambda\delta)^{t/\delta} \approx e^{-\lambda t}$$

Recap

- Intro to continuous random variables
- Learned how to describe the PDF and CDF of continuous r.v.s
- Expectation and Variance in the context of continuous r.v.s
- Joint density functions
  - Independence
  - Conditional densities
- Uniform continuous distribution
- Exponential distribution
  - Connection to Geometric distribution