Final this Friday, Aug 11th 6-9pm → all free response

↳ Berkeleytime (starts at 6:10pm)

→ DSP (even earlier exams, start at Berkeleytime)

Lecture 7A:

Continuous Probability III

Course Evals

↳ 80% 1 point extra credit post-curve

→ currently at 50%

Special Topics

↳ Wed (ECCs)

↳ Thurs (Computational

Stickers for CS70

UC Berkeley CS70

Summer 2023

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Exponential Distribution

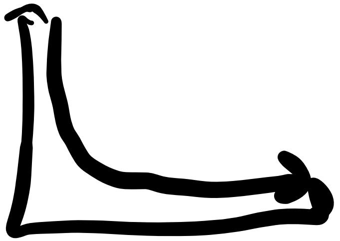
Def: For some λ > 0, a continuous random variable X is an exponential random variable with parameter λ if it has the following PDF,

$\lambda e^{-\lambda x}$ if $x \geq 0$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

We can write X ~ Exp(λ) if is an exponential random variable.

e^{-x}



Let's check that f(x) satisfies the two PDF properties

$\lambda > 0$

① $f_X(x) \geq 0 \rightarrow \lambda e^{-\lambda x} \geq 0$

positive

② $\int_{-\infty}^{\infty} f_X(x) = 1 = \int_{-\infty}^{\infty} \lambda e^{-\lambda x} dx = \int_{0}^{\infty} \lambda e^{-\lambda x} dx$

Expectation & Variance of Exponential

For a random variable X ~ Exp(λ),

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$\int e^{-y} dy$ ✓

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$y = \lambda x$

$y = \lambda x$

$$= \int_0^{\infty} x \lambda e^{-\lambda x} dx \cdot \frac{dy}{dx} \cdot \frac{1}{\lambda}$$

$$= \frac{1}{\lambda} \int_0^{\infty} y e^{-y} dy$$

$-e^{-\lambda x}\Big|_0^{\infty}$

$-e^{-\infty} - (-e^{-0})$

$\frac{1}{e^{\infty}} = 0$

$0 - (-e^{-0})$

$0 + 1$

$= 1$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{1}{\lambda}\left(-ye^{-y} - \int_0^\infty -e^{-y}\,dy\right)\Bigg|_0^\infty$$

$u = y \quad du = 1$

$dv = e^{-y}dy \quad v = -e^{-y}$

e^{-y}

$\downarrow$

$-e^{-y}$

$$= \frac{1}{\lambda}\left(-ye^{-y} - e^{-y}\right)\Big|_0^\infty$$

$$= \frac{1}{\lambda}\left(0 - 0 - \left(0 - e^{-0}\right)\right)$$

$$= \frac{1}{\lambda}\left(e^{0}\right) = \frac{1}{\lambda}(1) = \frac{1}{\lambda}$$

Expectation & Variance of Exponential

For a random variable X ~ Exp(λ), $\text{Var}(X) = \frac{1}{\lambda^2}.$

$$\text{Var}(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx$$

$y = \lambda x$

$\frac{dy}{dx} = \lambda$

$$= \frac{1}{\lambda}\int_0^\infty x^2\lambda e^{-y}\,dy \qquad y^2 = x^2\lambda^2$$

$$\cdot\lambda\cdot\frac{1}{\lambda}$$

$$= \frac{1}{\lambda^2}\int_0^\infty x^2\lambda^2 e^{-y}\,dy$$

$$= \frac{1}{\lambda^2}\underbrace{\int_0^\infty y^2 e^{-y}\,dy}$$

$$= \frac{1}{\lambda^2}(2) = \frac{2}{\lambda^2}$$

$$\mathrm{Var}(X) = \frac{2}{\lambda^2} - E(X)^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$ → exponential variance

Example Exponential

Going back to our terrible alarm clock, we know the behavior is:

Once plugged in, the alarm will randomly ring once after some amount of time, however we know it goes off at a rate of 1 time every 10 minutes.

Let X be the amount of time it takes the alarm to sound. X ~ Exp(1/10), because our rate of rings is 1/10 per minute.

How many minutes should we expect to wait before the alarm rings?

$$E(X) = \frac{1}{\text{parameter}} = \frac{1}{\left(\frac{1}{10}\right)} = 10 \text{ min}$$

An important property to note

$X \sim Exp(\lambda)$

$$\boxed{\mathbb{P}[X > t] = e^{-\lambda t}}$$

Proof:

$$P(X > t) = \int_{t}^{\infty} f_X(x)\,dx$$

$$= \int_{t}^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_{t}^{\infty} = -e^{-\infty} - (-e^{-\lambda t})$$

$$= 0 + e^{-\lambda t} = e^{-\lambda t}$$

Exponential Relation to Geometric

Let's try to draw a more rigorous connection between Exponential and Geometric distributions. Take our Geometric r.v. and consider running trials after every **δ** seconds. The probability of a success is p = λ**δ**.

Then, let Y denote the amount of time/seconds before a successful trial:

$$\mathbb{P}[Y > k\delta] = (1-p)^k = (1-\lambda\delta)^k,$$

To translate our trials to a continuum, consider taking the limit of d → 0. Then, for any time t,

$$\mathbb{P}[Y > t] = \mathbb{P}[Y > (\tfrac{t}{\delta})\delta] = (1-\lambda\delta)^{t/\delta} \approx e^{-\lambda t}$$

Continuous Example!

Let V and W have the joint density $12e^{-v-3w}$ for $0 < v < w < \infty$. Find the distribution of V.

$$f_{V,W}(v,w) = \begin{cases} 12e^{-v-3w} & \text{for } 0<v<w<\infty \\ 0 & \text{otherwise} \end{cases}$$

$$f_V(v) = \int f_{V,W}(v,w)\,dw$$

Continuous Example

$e^{-v-\infty} \to e^{-\infty}$

Let V and W have the joint density $12e^{-v-3w}$ for $0 < v < w < \infty$. Find the distribution of V.

$2 < w < v$

$0 \le v \le w < \infty$

$$= \int_v^\infty 12e^{-v-3w}\,dw = -4e^{-v-3w}\Big|_v^\infty$$

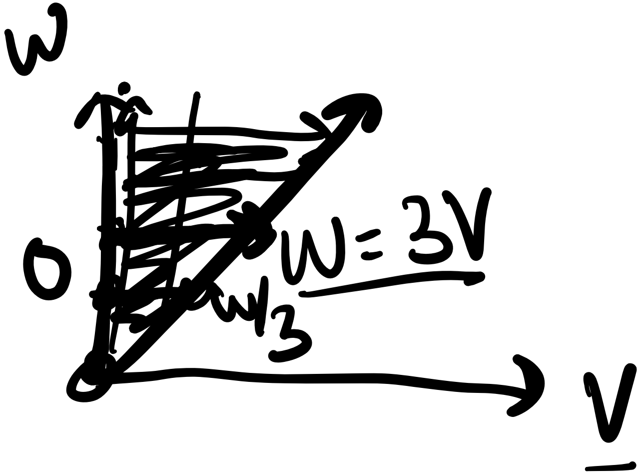
$$= \cancel{-4e^{-v-3\cdot\infty}}_0 - \left(-4e^{-v-3v}\right)$$

Continuous Example

$= 4e^{-4v}$ $\quad V \sim \text{Exp}(4)$

$\lambda = 4$

Let V and W have the joint density $12e^{-v-3w}$ for $0 < v < w < \infty$. Find $P(W > 3V)$



$W = 3V$

$v = \frac{w}{3}$

$W = 3V$

$W > 3V$

$0 \to \frac{w}{3}$

$$\int_0^\infty \int_0^{w/3} 12e^{-v-3w}\,dv\,dw$$

$$\int_0^\infty \left(\int_0^{w/3} 12e^{-v-3w}\,dv \right) dw$$

$$\downarrow$$

$$\int_0^\infty \left(-12e^{-v-3w} \Big|_0^{w/3} \right) dw$$

$$= \int_0^\infty \left(-12e^{-w/3-3w} - \left(-12e^{-0-3w}\right) \right) dw$$

$$\downarrow$$

$$\int_0^\infty \left(-12e^{-10/3w} + 12e^{-3w} \right) dw$$

Continuous Example

Let V and W have the joint density $12e^{-v-3w}$ for $0 < v < w < \infty$. Find $P(W > 3V)$

Another Continuous Example!

Let f(x) = k(2x+3) for -1 <= x <= 2, and 0 otherwise. What is k?

$$f_X(x) = \begin{cases} k(2x+3) & \underline{-1 \le x \le 2} \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

Another Continuous Example!

$2e^{-5x}$

Let f(x) = k(2x+3) for -1 <= x <= 2, and 0 otherwise. What is k?

$$\int_{-1}^{2} k(2x+3)\,dx = 1$$

$$\int_{-1}^{2} (2kx+3k)\,dx$$

$$\left.\left(kx^2+3xk\right)\right|_{-1}^{2}$$

$$=(4k+6k)-(k-3k)$$

$$=(10k)-(-2k)$$

$$=12k$$

$$12k=1, \quad k=\frac{1}{12}$$

Min of Uniforms

min > k
all numbers had to be > k

independent

We have n uniform(0, 2) random variables. For 0 < x < 2, what is the probability that the minimum of the RVs is <= x?

$$U_1, U_2, \ldots U_n \sim \text{Uniform}(0,2)$$

$$M = \min\{U_1, U_2, \ldots U_n\}$$

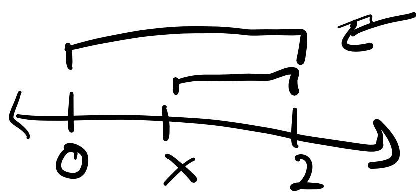
$$P(M \leq x) = 1 - P(M > x)$$

$$= 1 - P(U_1 \cap \ldots > x$$

$$= 1 - P(\underline{U_1 > x})\,P(\underline{U_2 > x}) \dots P(\underline{U_n > x})$$

$U_1 > x$

$\downarrow \quad U_1 : [0, 2]$



$$= \frac{2-x}{2-0} = \frac{2-x}{2}$$

$$= 1 - \left(\frac{2-x}{2}\right)^n$$

$= 1 \quad 1(\quad 1 \quad U_2 \quad U_n \quad)$

Min of Exponentials

independent

We have n exponential random variables with different parameters. Find the distribution of their minimum.

$\{\lambda_1, \lambda_2, \ldots \lambda_n\}$

$X_1 \sim Exp(\lambda_1) \quad X_2 \sim Exp(\lambda_2) \quad \ldots \quad X_n \sim Exp(\lambda_n)$

$M = \min\{X_1, X_2, \ldots X_n\}$

$P(X_1 > x) = e^{-\lambda_1 x}$

$P(X_2 > x) = e^{-\lambda_2 x}$

param. given

$P(M > x) = P(X_1 > x \cap X_2 > x \ldots X_n > x)$

$= P(X_1 > x) P(X_2 > x) \ldots P(X_n > x)$

$$= e^{-\lambda_1 x} e^{-\lambda_2 x} \cdots e^{-\lambda_n x}$$

$$= e^{-(\lambda_1 + \lambda_2 + \ldots \lambda_n) x}$$

$$M \sim \mathrm{Exp}(\lambda_1 + \lambda_2 + \ldots \lambda_n)$$

Formula from before

$$\boxed{P(X > t) = e^{-\lambda t}}$$

$X \sim \exp(2) \quad Y = 2X$

$$\boxed{P(X > t) = e^{-\lambda t}}$$

Transforming Exponentials

$Y = \exp\left(\frac{2}{2}\right) = \exp(1)$

Let X be exponential(λ) and let Y = 3X. What distribution does Y follow?

$X \sim \text{Exp}(\lambda) \qquad Y = 3X$

for any value t

$$P(Y > t) = P(3X > t) = P\left(X > \frac{t}{3}\right)$$

$$= 1 - P\left(X \le \frac{t}{3}\right)$$

plug in CDF

$4 \le X \le 5$

integer

decimal

4.18 4.65

Exponential

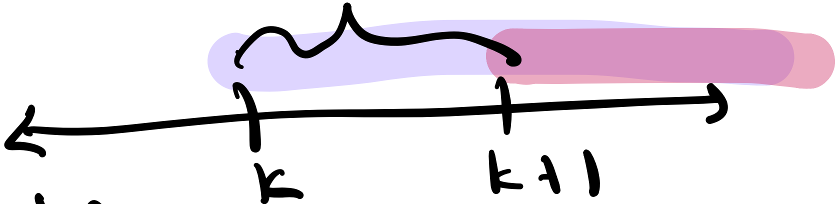
$T \sim \exp(1)$

$T = 4.375$, 5.623, 8.7686

Let T have the exponential(λ) distribution and let N be the integer part of T. What is the distribution of N?

$$P(N=k) = P(k < T < k+1)$$

$$= P(T > k) - P(T > k+1)$$



$$= e^{-\lambda k} - e^{-\lambda(k+1)}$$

Exponential (cont.)

$e^{-\lambda k} - e^{-\lambda k - \lambda}$

Let T have the exponential(λ) distribution and let N be the integer part of T. What is the distribution of N?

$= e^{-\lambda k}(1 - e^{-\lambda}) \rightarrow P(N = k+1)$

$= (e^{-\lambda})^{k}(1 - e^{-\lambda})$ ← geometric

$N \sim Geo(1 - e^{-\lambda})$

geometric but with values 0, 1, ...

T: 8.76

4.34

Combining Exponential and Uniform

2.4 $\rightarrow Y \sim \text{Uniform}(0, 2.4)$

Let X ~ Exp(λ) and let Y be Uniform [0, X]. What is the joint distribution of X and Y?

$X \sim \text{Exp}(\lambda)$

$\downarrow$

$f_X(x) = \lambda e^{-\lambda x}$

$Y \sim \text{Uniform}(0, X)$

$X = b$

$f_Y(y) = \frac{1}{x-0} = \frac{1}{b}$ $\quad 0 \le a \le b$

Combining Exponential and Uniform $X = x \to \frac{1}{x}$

Let X ~ Exp(λ) and let Y be Uniform [0, X]. What is the joint distribution of X and Y?

$$f_{X,Y}(x,y) = \underline{f_{Y|X}(y|x)}\,\underline{f_X(x)}$$

$$f_{X,Y}(x,y) = \frac{1}{x}\,\lambda e^{-\lambda x}$$

$\underline{0 \le y \le x}$

X doesn't take on negative values

Continuous Tail Sum (non-negative RV X)

$$E(X) = \sum_{k=0}^{\infty} P(X > k)$$

CDF

$$\mathbf{E}(X) = \int_0^\infty [1 - F(x)]dx = \boxed{\int_0^\infty \mathbf{P}(X > x)dx} .$$

→ HW problems

$$E(X) = \int_0^\infty x f_X(x)dx$$

$$t \Big|_0^x = x$$

$$= \int_0^\infty \int_0^x f_X(x)\,dt\,dx$$

$$= \int_0^\infty \boxed{\int_t^\infty f_X(x)dx}\,dt = \int_0^\infty P(X > t)dt$$

$$\int_t^\infty f_X(x)dx \rightarrow P(X > t)$$

Continuous Tail Sum (non-negative RV X)

Memoryless Exponentials

For X ~ Exp(λ) find P(X > s + n | X > n) $= P(X > s)$

$X \sim Exp(\lambda)$

$$P(X > s+n \mid X > n) = \frac{P(X > s+n \cap \cancel{X > n})}{P(X > n)}$$

$$= \frac{P(X > s+n)}{P(X > n)} = \frac{e^{-\lambda(s+n)}}{e^{-\lambda n}} = e^{-\lambda s}$$

$P(X > s)$

$P(X \geq S)$

Memoryless Exponentials

For X ~ Exp(λ) find P(X > s + t | X > t)

If you like probability :))

Upper-div → Stat134 → easier than both

→ Data140 → coding problems

→ EECS126 → theory heavy (random graphs, info theory)

$\underbrace{\text{Stat134, Data140, EECS126}}_{\text{assumes CS70}}$

Other classes → CS174: Randomized Algorithms

is pre-req: CS170 (algs)

→ CS270: Combinatorial Algorithms

Recap

Practiced Continuous R.V.s:

- Finding marginals from joint densities
- Finding constants of integration, E(X), Var(X) from PDF

More Properties of Exponential/Uniform:

- Min of Exponentials/Uniforms
- Memoryless Property of Exponential
- Continuous Tail Sum