UC Berkeley
Department of Electrical Engineering and Computer Sciences
EE126: Probability and Random Processes
Discussion 12
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There are three 'equivalent' definitions of a jointly Gaussian (JG) random vector.

- A random vector $X = (X_1, X_2, \ldots, X_k)^T$ is JG if there exists a base random vector $Z = (Z_1, Z_2, \ldots, Z_\ell)^T$, each of which component is an independent standard normal random variable, a transition matrix $A \in \mathbb{R}^{k \times \ell}$, and a mean vector $\mu \in \mathbb{R}^k$, such that $X = AZ + \mu$.
- A random vector $X = (X_1, X_2, \ldots, X_k)^T$ is JG if $Y = \sum_{i=1}^k a_i X_k$ is normally distributed for any $a = (a_1, a_2, \ldots, a_k)^T \in \mathbb{R}^k$. *Note: a point mass on a value is considered as a normal distribution with zero variance.* Thus, if $X_1 = \mathcal{N}(0,1)$ and $X_2 = -X_1$, $X = (X_1, X_2)^T$ still is a JG random vector.
- (Degenerate case only) A random vector $X = (X_1, X_2, \ldots, X_k)^T$ is JG if

$$f_X(x) = \frac{1}{\sqrt{|\Sigma|}(2\pi)^{n/2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right).$$

  for a nonnegative-definite matrix $\Sigma \in \mathbb{R}^{k \times k}$ and a mean vector $\mu \in \mathbb{R}^k$.

Some useful facts:

- If (X, Y) is JG, MMSE$[X|Y]$ = LLSE$[X|Y] = E[X|Y]$.
- MMSE$[X|Y]$ = LLSE$[X|Y]$ $\not\Longrightarrow$ (X, Y) is JG.
- X_1 and X_2 are marginally distributed as Gaussian $\not\Longrightarrow$ $X = (X_1, X_2)^T$ is JG.

Problem 1. (Fall 2015 Final) Let U, V be jointly Gaussian random variables with means $\mu_U = 1$, $\mu_V = 4$ and $\sigma_U^2 = 2.5, \sigma_V^2 = 2$ and covariance $\rho = 1$. Can we write $U = aV + Z$ where a is a scalar and Z is independent of V? If so, find a and Z, if not explain why.

Problem 2. (Fall 2015 Final) Assume that $(X, Y_n, n \geq 0)$ are mutually independent random variables with $X \sim N(0,1)$, $Y_n \sim N(0,1)$. Let $\hat{X}_n$ be the MMSE of X given $X + Y_1, X + Y_2, \ldots, X + Y_n$.

(a) Show that $\hat{X}_n = a_n(nX + \sum_{i=1}^n Y_i)$

(b) Find a_n.

Problem 3. (a) Consider zero-mean random variables X, Y, Z such that Y, Z are orthogonal. Show that $L[X|Y, Z] = L[X|Y] + L[X|Z]$.

(b) Show that for any random variables X, Y, Z it holds that:

$$L[X|Y, Z] = L[X|Y] + L[X|Z - L[Z|Y]]$$