UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Problem Set 5
Spring 2016

Issued: Thursday, February 25, 2016 **Due:** 9:00am Thursday, March 3, 2016

Problem 1. In order to estimate the probability of a head in a coin flip, p, you flip a coin n times and count the number of heads, S_n. You use the estimator $\hat{p} = \frac{S_n}{n}$.

(a) You choose the sample size n to have a guarantee

$$\Pr(|\hat{p} - p| \geq \epsilon) \leq \delta.$$

Using Chebyshev inequality, determine n with the following parameters:

(i) Compare the value of n when $\epsilon = 0.05$, $\delta = 0.1$ to when $\epsilon = 0.1, \delta = 0.1$.

(ii) Compare the value of n when $\epsilon = 0.1, \delta = 0.05$ to when $\epsilon = 0.1, \delta = 0.1$.

(b) Now, we change the scenario slightly. You know that $p \in (0.4, 0.6)$ and would now like to determine the smallest n such that:

$$\Pr(\frac{|\hat{p} - p|}{p} \leq 0.05) \geq 0.95$$

Problem 2. Let X_i, $1 \leq i \leq n$ be a sequence of i.i.d. random variables distributed uniformly in $[-1, 1]$. Show that the following sequences converge in probability to some limit.

(a) $Y_n = (X_n)^n$

(b) $Y_n = \prod_{i=1}^n X_i$

(c) $Y_n = \max\{X_1, X_2, \ldots, X_n\}$.

Problem 3. I break a stick n times in the following manner: the ith time I break the stick, I keep a fraction X_i of the remaining stick where X_i is uniform on the interval $[0, 1]$ and $X_1, X_2, \ldots, X_n$ are i.i.d. Let $P_n = \prod_{i=1}^n X_i$ be the fraction of the original stick that I end up with. Find $\lim_{n\to\infty} P_n^{1/n}$ and $E[P_n]^{1/n}$.

Problem 4. Consider the Markov chain of Figure 1, where $a, b \in (0, 1)$.

(a) Find the invariant distribution.

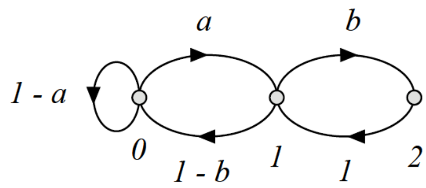

Figure 1: Markov Chain for Problem 4

(b) Calculate $\Pr(X(1) = 1, X(2) = 0, X(3) = 0, X(4) = 1 | X(0) = 0)$.

(c) Show that the Markov chain is aperiodic.

Problem 5. Let $\{X_n, n \geq 0\}$ be a Markov chain with two states, -1 and 1, and transition probabilities $P(-1, 1) = P(1, -1) = a$ for $a \in (0, 1)$. Define,

$$Y_n = X_0 + X_1 + \cdots + X_n.$$

Is $\{Y_n, n \geq 0\}$ a Markov chain? Prove or disprove.

Problem 6. You have a database of an infinite number of movies. Each movie has a rating that is uniformly distributed in $\{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5\}$ independent of all other movies. You want to find two movies such that the sum of their ratings is greater than 7.5 (7.5 is not included).

(a) A Stanford student chooses two movies each time and calculates the sum of their ratings. If it is less than or equal to 7.5, the student throws away these two movies and chooses two other movies. The student stops when he/she finds two movies such that the sum of their ratings is greater than 7.5. What is the expected number of movies that this student needs to choose from the database?

(b) A Berkeley student chooses movies from the database one by one and keeps the movie with the highest rating. The student stops when he/she finds that the sum of the ratings of the last movie that he/she has chosen and the movie with the highest rating among all the previous movies is greater than 7.5. What is the expected number of movies that the student will have to choose?

Now consider the ratings of movies can be real numbers and assume that the ratings are i.i.d. uniformly distributed in $[0, 5]$.

(c) What is the expected number of movies that the Stanford student will have to choose in order to find two movies such that the sum of their ratings is greater than 7.5?

(d) (Optional) What is the expected number of movies that the Berkeley student will have to choose in order to find two movies such that the sum of their ratings is greater than 7.5?