UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 8 (Optional)
Spring 2016

Issued: Thursday, March 17, 2016 **Due:** 11:59, Tuesday, March 29, 2016

Problem 1. Consider the continuous time Markov chain with state space $\{1, 2, 3, 4\}$ and the rate matrix

$$Q = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

(a) Find the stationary distribution π.

(b) Suppose the chain starts in state 1. What is the expected amount of time till it changes state for the first time?

(c) Again assume the chain starts in state 1. What is the expected amount of time till the chain is in state 4?

Problem 2. A queue has Poisson arrivals with rate λ. It has two servers that work in parallel. Where there are at least two customers in the queue, two are being served. When there is only one customer, only one server is active. The service times are i.i.d. Exp(μ).

(a) Argue that the queue length is a Markov Chain.

(b) Draw the state transition diagram.

(c) Find the minimum value of μ so that the queue is positive recurrent and solve the balance equations.

Problem 3. Consider a queue equipped with one server whose service rate μ. Jobs arrive to the queue according to a two state Markov Chain Y_t in the following way: when $Y_t = 0$, arrivals are Poisson with rate λ_0, when $Y_t = 1$, arrivals are Poisson with rate λ_1. Let X_t give the queue length at time t. Is X_t a Markov chain?

Problem 4. Empty taxis pass by a street corner at a Poisson rate of two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner at a Poisson rate of one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. John arrives at the street corner at a given time. Find his expected waiting time, given that he joins the queue. Assume that the process is in steady state.

Problem 5. A continuous-time queue has Poisson arrivals with rate λ, and it is equipped with infinitely many servers. The servers can work in parallel on multiple customers, but they are non-cooperative in the sense that a single customer can only be served by one server. Thus, when there are k customers in the queue, k servers are active. Suppose that the service time of each customer is exponentially distributed with rate μ and they are i.i.d.

(a) Argue that the queue-length is a Markov chain. Draw the transition diagram of the Markov chain.

(b) Prove that for all finite values of λ and μ the Markov chain is positive recurrent and find the invariant distribution.

Problem 6. Apple announces a surprise one-day promotion for brand-new iPhone 7: the price of a new phone will be 0 dollar at time 0, and it will increase by 5 dollars per minute! As soon as you wake up and check Twitter feeds (Huffman-coded), you run all the way to the nearest Apple store. Since there are too many Apple fanboys in your city, there are already too many customers waiting in line for service. There are 2 queues: the first queue has i customers including the one being served, and the second queue has j customers as depicted in Figure 1.
Assume that service time for each customer is exponentially distributed with a mean of 10 minutes. We also assume that the price is 0 at the moment you get to the Apple store. The price is proportional to the waiting time from your arrival to the moment that your service starts, but your service time is not included.

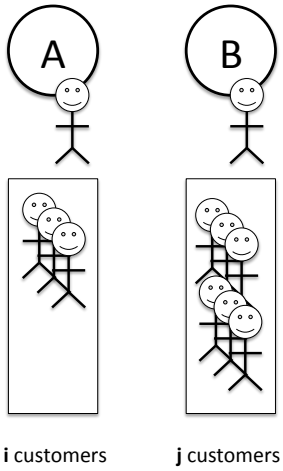

Figure 1: Apple Store on the 4th street, Berkeley

(a) Assume $i = j = 5$. You randomly choose one of the two queues and wait for service. What will be the expected price of the iPhone 6s when your service?

(b) Assume $i = 10,\ j = 11$. You need to choose one of the two queues. Which queue would you choose? You don't need to rigorously justify your choice but briefly explain why. What will be the expected price of the iPhone 6s when you start getting served?

(c) Assume $i = 10,\ j = 11$. Your younger brother suggests the following: he can wait for you in another queue so that if he reaches the server earlier than you, you can move to the other queue! He wants 50 dollars for helping you with this plan. Is it smart for you to hire him? In other words, would you expect a lower price for your iPhone 6s if you hired him?

(d) Assume $i = 10,\ j = 11$, and you hired your brother. What is the probability that your brother reaches the server earlier than you?