UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 12
Spring 2016

Issued: Thursday, April 21, 2016 **Due:** Thursday, April 28, 2016

Problem 1. Suppose that a hidden Markov model has hidden states X_1 and X_2. The states X_i can be either a or b, and the observations Y_i can be either 0 or 1. Suppose $P(X_1 = a) = 0.8$, $P(X_2 = a|X_1 = a) = 0.9$, $P(X_2 = a|X_1 = b) = 0.1$, $P(Y_i = 0|X_i = a) = 0.9$, and $P(Y_i = 0|X_i = b) = 0.2$. Suppose you have the observation sequence $Y_1 = 0$, $Y_2 = 1$.

(a) For both X_1 and X_2, compute $P(X_i|Y_1, Y_2)$, and write down the prediction if we choose X_i to maximize this marginal probability.

(b) Suppose instead we maximize $P(X_1, X_2|Y_1, Y_2)$ over X_1 and X_2 jointly. What is the prediction for X_1 and X_2 in this case?

Problem 2. Assume that the Markov chain $\{X_n, n \geq 0\}$ with states 0 and 1, and initial distribution $\pi_0(0) = \pi_0(1) = 0.5$ and $P(x, x') = 0.3$ for $x \neq x'$ and $P(x, x) = 0.7$ $(x, x' \in \{0, 1\})$. Assume also that X_n is observed through a BSC with error probability 0.1. The observations are denoted by Y_n. Suppose the observations are $(Y_0, \ldots, Y_4) = (0, 0, 1, 1, 1)$. Use the Viterbi algorithm to find the most likely sequence of the states $(X_0, \ldots, X_4)$.

Problem 3. Let $(X, Y, Z)^T \sim N(\mu, \Sigma)$, and

$$\mu = [0, 0, 0]^T,$$

and

$$\Sigma = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 2 \end{bmatrix}.$$

Find $E[X|Y, Z]$.

Problem 4. Let $Y = X + Z$ and $U = X - Z$, where X and Z are IID $N(0, 1)$.

(a) Find the joint distribution of U and Y.

(b) Find the MMSE of X given the observation Y, call this $\hat{X}(Y)$.

(c) Let the estimation error $E = X - \hat{X}(Y)$. Find the conditional distribution of E given Y.

Problem 5. Forrest Gump is running across the United States, and we would like to track his progress. Assume that on day n he runs $X(n)$ miles, and the amount he runs each day is determined by the amount he ran on the previous day with some random noise in the following manner: $X(n) = \alpha X(n-1) + V(n)$. Unfortunately, the measurements of the distance he traveled on each day are also subject to some noise. Assume that $Y(n)$ gives the measured number of miles Forrest Gump traveled on day n and that $Y(n) = \beta X(n) + W(n)$. For this problem, assume that $X(0) \sim N(0, \sigma_X^2), W(n) \sim N(0, \sigma_W^2), V(n) \sim N(0, \sigma_V^2)$ are independent.

(a) Suppose that you observe $Y(0)$. Find the MMSE of $X(0)$ given this observation.

(b) Express both $E[Y(n)|Y(0), \ldots, Y(n-1)]$ and $E[X(n)|Y(0), \ldots, Y(n-1)]$ in terms of $\hat{X}(n-1)$, where $\hat{X}(n-1)$ is the MMSE of $X(n-1)$ given the observations $Y(0), Y(1), \ldots, Y(n-1)$.

(c) Show that:

$$\hat{X}(n) = \alpha \hat{X}(n-1) + k_n[Y(n) - \alpha\beta\hat{X}(n-1)]$$

where $k_n = \frac{\text{cov}(X(n), \tilde{Y}(n))}{\text{Var}(\tilde{Y}(n))}$ and $\tilde{Y}(n) = Y(n) - L[Y(n)|Y(0), Y(1), \ldots, Y(n-1)]$.

Hint: Think geometrically