EECS 127/227AT Optimization Models in Engineering UC Berkeley Fall 2022

Final

PRINT your student ID: ____________________

PRINT AND SIGN your name: __________, __________ __________
(last) (first) (sign)

PRINT your discussion sections and (u)GSIs (the ones you attend): __________

Name and SID of the person to your left: __________

Name and SID of the person to your right: __________

Name and SID of the person in front of you: __________

Name and SID of the person behind you: __________

1. **Honor Code (0 pts)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   If you do not copy the honor code and sign your name, you will get a 0 on the exam.

2. **SID (3 pts)**

   When the exam starts, write your SID at the top of every page.

   No extra time will be given to complete this task.

3. **Favorites. Any answer, as long as you write it down, will be given full credit. (2 pts)**

   (a) (1 pts) What's something that made you happy this year?

   (b) (1 pts) What's your favorite number?

Do not turn the page until your proctor tells you to do so.

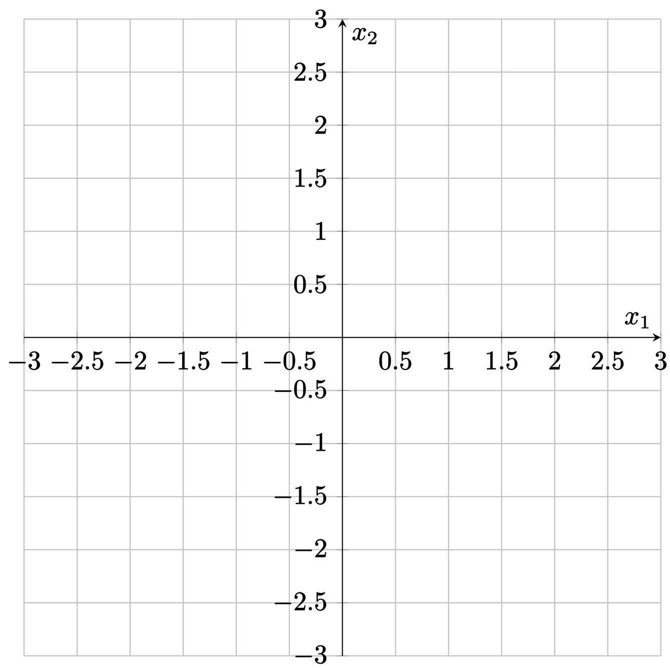
4. Linear Program (12 pts)

Consider the linear program

$$\begin{aligned} \min_{\vec{x} \in \mathbb{R}^2} \quad & \begin{bmatrix} 1 \\ -1 \end{bmatrix}^\top \vec{x} \\ \text{s.t.} \quad & \begin{bmatrix} -1 \\ -1 \end{bmatrix} \leq \vec{x} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ & 0 \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}^\top \vec{x} \leq 1.5. \end{aligned} \tag{1}$$

where $\vec{x} \doteq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

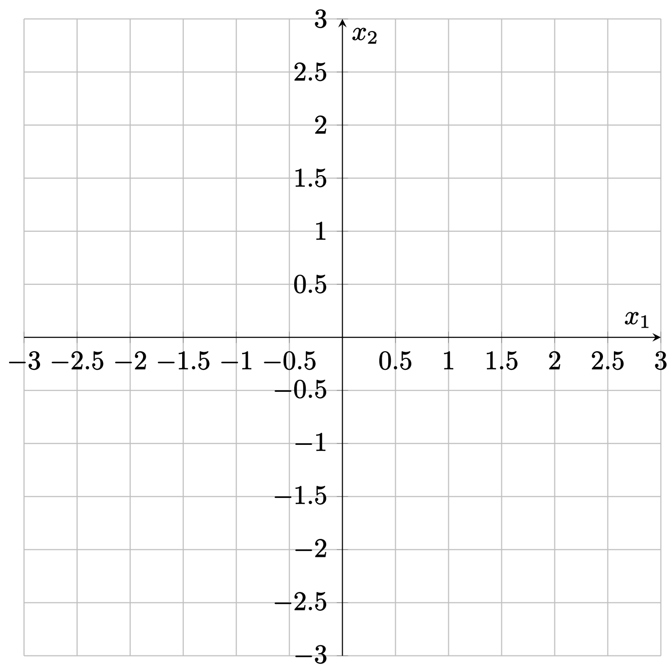
(a) (6 pts) **Draw the constraints on this problem and shade in the feasible region.**



Recall the linear program (1):

$$\begin{aligned} \min_{\vec{x}\in\mathbb{R}^2} \quad & \begin{bmatrix} 1 \\ -1 \end{bmatrix}^\top \vec{x} \\ \text{s.t.} \quad & \begin{bmatrix} -1 \\ -1 \end{bmatrix} \leq \vec{x} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ & 0 \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}^\top \vec{x} \leq 1.5. \end{aligned} \tag{1}$$

(b) (3 pts) **Plot and label level sets of the objective**, i.e., $\begin{bmatrix} 1 \\ -1 \end{bmatrix}^\top \vec{x} = k$ for $k = \{-2, 0, 2\}$ on the figure below.



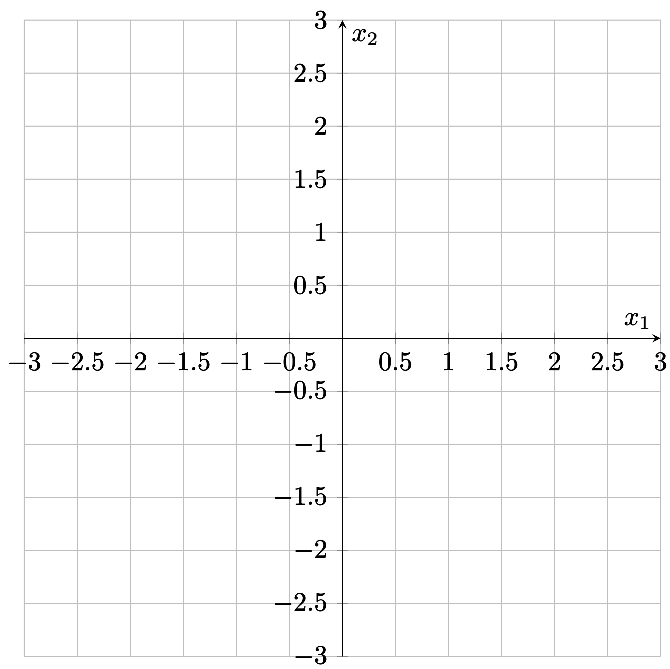
Recall the linear program (1):

$$\begin{aligned} \min_{\vec{x} \in \mathbb{R}^2} \quad & \begin{bmatrix} 1 \\ -1 \end{bmatrix}^\top \vec{x} \\ \text{s.t.} \quad & \begin{bmatrix} -1 \\ -1 \end{bmatrix} \leq \vec{x} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ & 0 \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}^\top \vec{x} \leq 1.5. \end{aligned} \tag{1}$$

(c) (3 pts) **Identify the optimal value $p^\star$ for the problem (1) and the vector $\vec{x}^\star$ which achieves it**. *You do not need to justify your answer.*

What are the active constraints at the optimal solution? *You do not need to justify your answer.*

NOTE: It may be helpful to draw the level sets and constraints on one plot. This plot below will not be graded; it is just there for your convenience.



5. Weak vs Strong Duality (13 pts)

Consider the convex problem

$$p^\star = \min_{\vec{x} \in \mathbb{R}^2} \quad \frac{1}{2}(x_1 + 1)^2 + x_2^2 \qquad (2)$$
$$\text{s.t.} \quad x_1 = 0.$$

(a) (2 pts) **Find the primal optimum $p^\star$ in problem (2) by substituting the constraint $x_1 = 0$ into the objective function.** *You do not need to justify your answer.*

(b) (3 pts) **Does Slater's condition hold for problem (2)? Does strong duality hold?** *Justify your answer.*

Recall problem (2):

$$p^\star = \min_{\vec{x}\in\mathbb{R}^2} \quad \frac{1}{2}(x_1+1)^2 + x_2^2 \qquad (2)$$
$$\text{s.t.} \quad x_1 = 0.$$

(c) (8 pts) **Find the dual function $g(\nu)$ and the dual optimum $d^\star = \max_{\nu\in\mathbb{R}} g(\nu)$.** *Show your work.*

[Extra page for scratch work that will not be graded unless you tell us in the original problem space.]

6. Transformations (12 pts)

For each of the below problems, assume:

- $\vec{x} \in \mathbb{R}^n$;
- $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$;
- $\mathcal{X} \subset \mathbb{R}^n$.

Shade in or circle "True" if the statement is **always** true. Otherwise, shade in or circle "False". *Ensure that the option you select is clear. You do not need to justify your answer. No partial credit will be awarded.*

(a) (3 pts) Suppose $\max_{\vec{x} \in \mathcal{X}} f(\vec{x}) < \infty$.

$$\max_{\vec{x} \in \mathcal{X}} f(\vec{x}) = -\left[\min_{\vec{x} \in \mathcal{X}} -f(\vec{x})\right]. \tag{3}$$

○ True

○ False

(b) (3 pts) Suppose $\Omega \subseteq \mathcal{X}$, i.e., Ω is a subset of $\mathcal{X}$.

$$\max_{\vec{x} \in \mathcal{X}} f(\vec{x}) \leq \max_{\vec{x} \in \Omega} f(\vec{x}). \tag{4}$$

○ True

○ False

(c) (3 pts) Suppose $\max_{\vec{x} \in \mathcal{X}} f(\vec{x}) < \infty$, $\max_{\vec{x} \in \mathcal{X}} g(\vec{x}) < \infty$, and both maxima are achieved.

$$\max_{\vec{x} \in \mathcal{X}} \left[f(\vec{x}) + g(\vec{x})\right] \leq \max_{\vec{x} \in \mathcal{X}} f(\vec{x}) + \max_{\vec{x} \in \mathcal{X}} g(\vec{x}). \tag{5}$$

○ True

○ False

(d) (3 pts) Suppose $\max_{\vec{x} \in \mathcal{X}} f(\vec{x}) < \infty$ and the maximum is achieved at a unique maximizer.

$$\operatorname*{argmax}_{\vec{x} \in \mathcal{X}} e^{f(\vec{x})} = \operatorname*{argmax}_{\vec{x} \in \mathcal{X}} f(\vec{x}). \tag{6}$$

○ True

○ False

[Extra page for scratch work that will not be graded unless you tell us in the original problem space.]

7. Low Rank Approximation (3 pts)

Let $A \in \mathbb{R}^{3\times 4}$ be a matrix whose full SVD is

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}}_{V^\top}. \tag{7}$$

Give the best rank-1 approximation to A, i.e., the solution to the problem

$$\underset{\substack{B \in \mathbb{R}^{3\times 4} \\ \mathrm{rk}(B) \leq 1}}{\operatorname{argmin}} \|A - B\|_F^2 . \tag{8}$$

No justification is necessary. No partial credit will be awarded.

NOTE: Please leave your answer in terms of a matrix product.

8. SOCP (12 pts)

Consider a matrix $A \in \mathbb{R}^{m \times n}$ and vectors $\vec{b} \in \mathbb{R}^m$, $\vec{c} \in \mathbb{R}^n$ and scalar $d \in \mathbb{R}$. Consider the problem

$$\min_{\vec{z} \in \mathbb{R}^n} (\|A\vec{z} - \vec{b}\|_2 - \vec{c}^\top \vec{z} - d)^2. \tag{9}$$

(a) (8 pts) Suppose $m = 1$ and $n = 1$. Then $\vec{z} = z$ is just a scalar, and $A, \vec{b}, \vec{c}$ are also just scalars. In particular, suppose $A = 1$, $\vec{b} = 1$, $\vec{c} = 1$, and $d = 1$. **For these values, is the optimization problem (9) convex?** *Justify your answer.*

HINT: First, rewrite the problem with the given values. Then, consider evaluating the objective function at $z = 0$ and $z = 2$.

(b) (4 pts) The problem can be reformulated as

$$\min_{\vec{x}\in\mathbb{R}^{n+1}} \quad \begin{bmatrix}\vec{0}\\ 1\end{bmatrix}^\top \vec{x} \tag{10}$$

$$\text{s.t.} \quad \left\|\begin{bmatrix}A & \vec{0}\end{bmatrix}\vec{x} - \vec{b}\right\|_2 - \begin{bmatrix}\vec{c}\\ 1\end{bmatrix}^\top \vec{x} - d \leq 0 \tag{11}$$

$$\left\|\begin{bmatrix}A & \vec{0}\end{bmatrix}\vec{x} - \vec{b}\right\|_2 - \begin{bmatrix}\vec{c}\\ -1\end{bmatrix}^\top \vec{x} - d \geq 0. \tag{12}$$

where $\vec{0}$ is the all-zeros vector of the appropriate dimension. **Which constraint should be dropped to make the problem an SOCP?** *Justify your answer.*

9. Power Purchase (20 pts)

This problem describes a simplified version of the optimization problem on the power grid. Consider a consumer who wants to consume d units (in kilo-Watt-hour (kWh)) of electrical energy. The consumer can purchase this energy from a combination of electricity generators. Consider n generators indexed by $i = 1, 2, ..., n$, and denote the energy output of the i^{th} generator by x_i. The cost of generating x_i units of energy is given by a generator specific cost $f_i(x_i)$:

$$f_i(x_i) = a_i x_i^2 + b_i x_i \tag{13}$$

where $a_i \geq 0$. Each generator has a constraint on the maximum energy it can produce, and this cap is denoted by m_i, i.e., $0 \leq x_i \leq m_i$.

The consumer tries to purchase d units of energy at minimum cost by **optimizing the amount of energy** x_i purchased from each generator by solving:

$$\begin{aligned} \min_{x_1, x_2, \ldots, x_n} \quad & \sum_{i=1}^{n} f_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = d \\ & 0 \leq x_i \leq m_i \quad i = 1, ..., n. \end{aligned} \tag{14}$$

(a) (3 pts) **Choose the smallest class that problem (14) belongs to (LP/QP/SOCP/etc.).** *You do not need to justify your answer.*

(b) (6 pts) We consider a specific case of the optimization problem (14):

$$\begin{aligned} \min_{x_1, x_2} \quad & \left(x_1^2 + 2x_1\right) + \left(\frac{3}{2}x_2^2\right) \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & 0 \le x_1 \le 2 \\ & 0 \le x_2 \le 1. \end{aligned} \tag{15}$$

This problem is a specific instance of (14) with $n = 2$, $d = 1$, $m_1 = 2$, and $m_2 = 1$. **What are the optimal values of x_1, x_2?** *Show your work.*

HINT: Try eliminating x_2 by replacing it with $1 - x_1$, solve the unconstrained optimization problem and check back to see if the constraints are satisfied.

(c) (6 pts) Consider a similar problem as in the previous subpart (b), this time with $d = 2$:

$$\begin{aligned} \min_{x_1, x_2} \quad & \left(x_1^2 + 2x_1\right) + \left(\frac{3}{2}x_2^2\right) \\ \text{s.t.} \quad & x_1 + x_2 = 2 \\ & 0 \le x_1 \le 2 \\ & 0 \le x_2 \le 1. \end{aligned} \tag{16}$$

What are the optimal values of x_1, x_2? *Show your work.*

HINT: Try eliminating x_2 by replacing it with $2 - x_1$. Check the function value at the critical points of the problem, i.e., the points where the gradient is zero or undefined, points on the boundaries, and $\pm\infty$.

(d) (5 pts) Consider an instance of (14) where $n = 2$, two generators generate energy x_1, x_2 to fulfill demand d, with associated costs $f_1(x_1)$ and $f_2(x_2)$ and capacities m_1, m_2:

$$\begin{aligned} \min_{x_1, x_2} \quad & f_1(x_1) + f_2(x_2) \\ \text{s.t.} \quad & x_1 + x_2 = d \\ & 0 \le x_i \le m_i, \quad i = 1, 2. \end{aligned} \tag{17}$$

Consider the dual variables corresponding to the constraints as below:

Dual Variable	Constraint
ν	$x_1 + x_2 = d$
λ_1	$0 \le x_1$
λ_2	$x_1 \le m_1$
λ_3	$0 \le x_2$
λ_4	$x_2 \le m_2$

Write the complementary slackness conditions for this problem.

Suppose you solve the above problem (17), and find that your solution is such that $0 < x_1^\star < m_1$ and $0 < x_2^\star = m_2$. **Which dual variables are necessarily equal to zero?**

You may assume (without needing to prove) that strong duality holds.

[Extra page for scratch work that will not be graded unless you tell us in the original problem space.]

10. Proximal Operator (8 pts)

For a function $h\colon \mathbb{R}^n \to \mathbb{R}$, define its *proximal operator* $\mathsf{prox}_h\colon \mathbb{R}^n \to \mathbb{R}^n$ as

$$\mathsf{prox}_h(\vec{x}) \doteq \operatorname*{argmin}_{\vec{u}\in\mathbb{R}^n} \left(h(\vec{u}) + \frac{1}{2}\left\|\vec{x}-\vec{u}\right\|_2^2\right). \tag{18}$$

Consider a special case where $n = 1$, i.e., we have a scalar problem, and $h(x) \doteq \lambda\,|x|$ for a constant $\lambda \geq 0$. Let $x_0 \in \mathbb{R}$. Prove that

$$\mathsf{prox}_h(x_0) \doteq \operatorname*{argmin}_{u\in\mathbb{R}} \left(\lambda\,|u| + \frac{1}{2}(x_0-u)^2\right) \tag{19}$$

is the soft-thresholding function that we've seen in the context of the LASSO problem, i.e.,

$$\operatorname*{argmin}_{u\in\mathbb{R}} \left(\lambda\,|u| + \frac{1}{2}(x_0-u)^2\right) = \begin{cases} x_0+\lambda, & \text{if } x_0 < -\lambda \\ 0, & \text{if } -\lambda \leq x_0 \leq \lambda \\ x_0-\lambda, & \text{if } x_0 > \lambda. \end{cases} \tag{20}$$

[Extra page for scratch work that will not be graded unless you tell us in the original problem space.]

11. Modified SVM (31 pts)

Let $Z \in \mathbb{R}^{n\times d}$ be a constant known matrix, and let $C > 0$ be a scalar. Consider the problem

$$p^\star = \min_{\vec{w},\vec{s}} \quad \frac{1}{2}\|\vec{w}\|_2^2 + \frac{C}{2}\|\vec{s}\|_2^2 \qquad (21)$$
$$\text{s.t.} \quad \vec{s} \geq \vec{0}$$
$$\vec{s} \geq \vec{1} - Z\vec{w},$$

where $\vec{w} \in \mathbb{R}^d$ and $\vec{s} \in \mathbb{R}^n$ are the optimization variables, $\vec{0} = \begin{bmatrix}0 & \cdots & 0\end{bmatrix}^\top \in \mathbb{R}^n$ is the vector of all zeros, $\vec{1} = \begin{bmatrix}1 & \cdots & 1\end{bmatrix}^\top \in \mathbb{R}^n$ is the vector of all ones. Strong duality holds for this problem.

(a) (3 pts) **Choose the smallest class that problem (21) belongs to (LP/QP/SOCP/etc).** *You do not need to justify your answer.*

(b) (4 pts) **Are the KKT conditions for problem (21) necessary, sufficient or both necessary and sufficient for global optimality?**

NOTE: You may use (without needing to prove) the fact that strong duality holds.

Recall problem (21):

$$
\begin{aligned}
p^\star = \min_{\vec{w},\vec{s}} \quad & \frac{1}{2}\|\vec{w}\|_2^2 + \frac{C}{2}\|\vec{s}\|_2^2 \\
\text{s.t.} \quad & \vec{s} \geq \vec{0} \\
& \vec{s} \geq \vec{1} - Z\vec{w},
\end{aligned} \tag{21}
$$

where $\vec{w} \in \mathbb{R}^d$ and $\vec{s} \in \mathbb{R}^n$.

(c) (2 pts) **Let $\vec{\alpha}$ be the dual variable corresponding to the constraint $\vec{s} \geq \vec{0}$. What is the dimension (i.e., number of entries) of $\vec{\alpha}$?** *You do not need to justify your answer.*

(d) (4 pts) **Show that the Lagrangian $L(\vec{w}, \vec{s}, \vec{\alpha}, \vec{\beta})$ of problem (21),** where $\vec{\alpha}$ is the dual variable corresponding to the constraint $\vec{s} \geq \vec{0}$, and $\vec{\beta}$ is the dual variable corresponding to the constraint $\vec{s} \geq \vec{1} - Z\vec{w}$, **is equal to**

$$
L(\vec{w}, \vec{s}, \vec{\alpha}, \vec{\beta}) = \frac{1}{2}\|\vec{w}\|_2^2 + \frac{C}{2}\|\vec{s}\|_2^2 - \vec{s}^\top(\vec{\alpha} + \vec{\beta}) - \vec{w}^\top Z^\top \vec{\beta} + \vec{1}^\top \vec{\beta}. \tag{22}
$$

Recall problem (21):

$$
\begin{aligned}
p^\star = \min_{\vec{w},\vec{s}} \quad & \frac{1}{2}\|\vec{w}\|_2^2 + \frac{C}{2}\|\vec{s}\|_2^2 \\
\text{s.t.} \quad & \vec{s} \geq \vec{0} \\
& \vec{s} \geq \vec{1} - Z\vec{w}.
\end{aligned} \tag{21}
$$

Recall that this problem has the Lagrangian

$$
L(\vec{w}, \vec{s}, \vec{\alpha}, \vec{\beta}) = \frac{1}{2}\|\vec{w}\|_2^2 + \frac{C}{2}\|\vec{s}\|_2^2 - \vec{s}^\top(\vec{\alpha} + \vec{\beta}) - \vec{w}^\top Z^\top \vec{\beta} + \vec{1}^\top \vec{\beta}. \tag{22}
$$

(e) (8 pts) **Write the KKT conditions for problem (21). Show that if $(\vec{w}^\star, \vec{s}^\star, \vec{\alpha}^\star, \vec{\beta}^\star)$ obey the KKT conditions for problem (21), then**

$$
\vec{w}^\star = Z^\top \vec{\beta}^\star \qquad \text{and} \qquad \vec{s}^\star = \frac{\vec{\alpha}^\star + \vec{\beta}^\star}{C}. \tag{23}
$$

HINT: For the first order/stationarity condition on the Lagrangian you will need to consider partial derivatives with respect to both $\vec{w}$ and $\vec{s}$.

[Extra page for scratch work that will not be graded unless you tell us in the original problem space.]

Recall problem (21):

$$\begin{aligned} p^\star = \min_{\vec{w},\vec{s}} \quad & \frac{1}{2}\|\vec{w}\|_2^2 + \frac{C}{2}\|\vec{s}\|_2^2 \\ \text{s.t.} \quad & \vec{s} \geq \vec{0} \\ & \vec{s} \geq \vec{1} - Z\vec{w}. \end{aligned} \tag{21}$$

Recall that this problem has the Lagrangian

$$L(\vec{w}, \vec{s}, \vec{\alpha}, \vec{\beta}) = \frac{1}{2}\|\vec{w}\|_2^2 + \frac{C}{2}\|\vec{s}\|_2^2 - \vec{s}^\top(\vec{\alpha} + \vec{\beta}) - \vec{w}^\top Z^\top \vec{\beta} + \vec{1}^\top \vec{\beta}. \tag{22}$$

(f) (5 pts) **Compute the dual function of problem (21) as**

$$g(\vec{\alpha}, \vec{\beta}) \doteq L(\vec{w}^\star(\vec{\alpha}, \vec{\beta}), \vec{s}^\star(\vec{\alpha}, \vec{\beta}), \vec{\alpha}, \vec{\beta}) \tag{24}$$

where from the previous part we have that

$$\vec{w}^\star(\vec{\alpha}, \vec{\beta}) = Z^\top \vec{\beta} \qquad \text{and} \qquad \vec{s}^\star(\vec{\alpha}, \vec{\beta}) = \frac{\vec{\alpha} + \vec{\beta}}{C}. \tag{25}$$

Your final expression for $g(\vec{\alpha}, \vec{\beta})$ should not contain any maximizations, minimizations or terms including $\vec{w}$, $\vec{s}$, $\vec{w}^\star$, or $\vec{s}^\star$. It should only contain $\vec{\alpha}$, $\vec{\beta}$, C, Z, and numerical constants. *Show your work.*

(g) (5 pts) Let $\vec{\alpha}^\star$ and $\vec{\beta}^\star$ be optimal dual variables that solve the problem

$$d^\star \doteq \max_{\vec{\alpha},\vec{\beta}\geq\vec{0}} g(\vec{\alpha},\vec{\beta}). \tag{26}$$

It turns out that $\vec{\alpha}^\star$ can also be obtained by solving the quadratic program:

$$\begin{aligned} \min_{\vec{\alpha}} \quad & \left\|\vec{\alpha}+\vec{\beta}^\star\right\|_2^2 \\ \text{s.t.} \quad & \vec{\alpha}\geq\vec{0}. \end{aligned} \tag{27}$$

Solve this quadratic program (27) directly and find $\vec{\alpha}^\star$. *Show your work.*

HINT: The duality or KKT approaches are not recommended. Consider $\vec{\alpha} = \begin{bmatrix}\alpha_1 & \cdots & \alpha_n\end{bmatrix}^\top$*, and use the components of* $\vec{\alpha}$ *to decompose the problem into* n *separate scalar problems. Solve each one by checking critical points; that is, points where the gradient is* 0*, the boundary of the feasible set, and* $\pm\infty$*.*

[Extra page for scratch work that will not be graded unless you tell us in the original problem space.]

12. Levenberg-Marquardt Regularization for Newton's Method (26 pts)

Newton's method often suffers from non-invertibility of the Hessian $H(\vec{x}) = \nabla^2 f(\vec{x})$. One solution is to use the modified Hessian $H(\vec{x}) + \mu I$, which is known as the *Levenberg-Marquardt regularized Hessian.* In this problem, we explore how to find an appropriate μ given certain conditions on f. This problem doesn't depend on an understanding of Newton's method, other than the first part.

(a) (3 pts) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be twice differentiable with Hessian $H(\vec{x})$. Let $\vec{x}_k$ be the k^{th} iterate of Newton's method on f. **Write the Newton's method step for $\vec{x}_{k+1}$ in terms of $\vec{x}_k$.** You may assume $H(\vec{x}_k)$ is invertible.

(b) (5 pts) **Show that for any symmetric matrix $A \in \mathbb{S}^n$ with $\lambda_i\{A\}$ as the i^{th} largest eigenvalue of A:**

$$\|A\|_F^2 = \sum_{i=1}^{n} \lambda_i\{A\}^2. \tag{28}$$

HINT: Consider using the eigendecomposition or SVD of A.

(c) (5 pts) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be twice differentiable with Hessian $H(\vec{x})$. Let $M > 0$ be such that $\|H(\vec{x})\|_F^2 \leq M^2$ for all $\vec{x} \in \mathbb{R}^n$. **Use the result in part (b) to show that**

$$-M \leq \lambda_{\min}\{H(\vec{x})\} \qquad \text{for all } \vec{x} \in \mathbb{R}^n, \tag{29}$$

where we let $\lambda_{\min}\{H(\vec{x})\}$ denote the smallest eigenvalue of the symmetric Hessian matrix $H(\vec{x})$. *HINT:* $\lambda_{\min}\{H(\vec{x})\}^2 \leq \sum_{i=1}^{n} \lambda_i\{H(\vec{x})\}^2$.

(d) (5 pts) Recall that $f: \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable function with Hessian $H(\vec{x})$. Even though $H(\vec{x})$ may not be positive definite, we would like to find a small $\mu \geq 0$ such that $H(\vec{x}) + \mu I$ is positive definite for all $\vec{x} \in \mathbb{R}^n$. Fix $\epsilon > 0$ and define the optimization problem:

$$\mu^\star = \min_{\mu \geq 0} \quad \mu \tag{30}$$
$$\text{s.t.} \quad \lambda_i\{H(\vec{x}) + \mu I\} \geq \epsilon, \quad \text{for all } i \in \{1, \ldots, n\} \text{ and } \vec{x} \in \mathbb{R}^n.$$

Show that the constraints in the above problem (30) are equivalent to

$$\lambda_{\min}\{H(\vec{x})\} \geq -\mu + \epsilon, \qquad \text{for all } \vec{x} \in \mathbb{R}^n. \tag{31}$$

(e) (8 pts) Recall that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable function with Hessian $H(\vec{x})$, and $M > 0$ is such that $\|H(\vec{x})\|_F^2 \leq M^2$ for all $\vec{x} \in \mathbb{R}^n$. From part (d), we consider the optimization problem with $\epsilon > 0$:

$$\begin{aligned} \mu^\star = \min_{\mu \geq 0} \quad & \mu \\ \text{s.t.} \quad & \lambda_{\min}\{H(\vec{x})\} \geq -\mu + \epsilon, \quad \text{for all } \vec{x} \in \mathbb{R}^n. \end{aligned} \tag{32}$$

In class, you have seen how using slack variables can create an equivalent program:

$$\begin{aligned} \max_{\vec{x} \in \mathcal{X}} f(\vec{x}) \quad = \quad \min_{c \in \mathbb{R}} \quad & c \\ \text{s.t.} \quad & f(\vec{x}) \leq c \quad \text{for all } \vec{x} \in \mathcal{X}. \end{aligned} \tag{33}$$

Using this equivalence between formulations, solve for $\mu^\star$. You may assume that there exists $\vec{x}_0 \in \mathbb{R}^n$ such that $\lambda_{\min}\{H(\vec{x}_0)\} = -M$, i.e., the lower bound in part (c) is achieved with equality at some point $\vec{x}_0$.

[Extra page for scratch work that will not be graded unless you tell us in the original problem space.]

[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.]

[Extra page for scratch work that will not be graded unless you tell us in the original problem space.]

Final Instructions

Read the following instructions before the exam.

There are 12 problems of varying numbers of points. You have 180 minutes for the exam. The problems are of varying difficulty, so pace yourself accordingly, do easier problems first, and avoid spending too much time on any one question until you have gotten all of the other points you can. Problems are not necessarily ordered in terms of difficulty, so be sure to read all the problems.

There are 34 pages on the exam, so there should be 17 sheets of paper in the exam. The exam is printed double-sided. Do not forget the problems on the back sides of the pages! Notify a proctor immediately if a page is missing. **Do not tear out or remove any of the pages. Do not remove the exam from the exam room.**

No collaboration is allowed, and do not attempt to cheat in any way. Cheating will not be tolerated.

Write your student ID on each page. If a page is found without a student ID, and some pages from your exam go missing, we will have no way of giving you credit for those pages. All exam pages will be separated during scanning.

You may consult TWO handwritten 8.5” × 11” note sheet(s) (front and back). No phones, calculators, tablets, computers, other electronic devices, or scratch paper are allowed.

Please write your answers legibly in the boxed spaces provided on the exam. The space provided should be adequate. **If you still run out of space, please use a blank page and clearly tell us in the original problem space where to look for your solution.**

Unless otherwise specified, show all of your work in order to receive full credit. Partial credit will be given for substantial progress on each problem.

If you need to use the restrooms during the exam, bring your student ID card, your phone, and your exam to a proctor. You can collect them once you return from the restrooms.

Our advice to you: if you can’t solve the problem, state and solve a simpler one that captures at least some of its essence. You might get some partial credit, and more importantly, you will perhaps find yourself on a path to the solution.

Good luck!

Do not turn the page until your proctor tells you to do so.