EE C128 / ME C134 - Feedback Control Systems

Lecture - Chapter 2 - Modeling in the Frequency Domain

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Lecture abstract

Topics covered in this presentation

- ► Laplace transform
- ► Transfer function
- Conversion between systems in time-, frequency-domain, and transfer function representations
- ► Electrical, translational-, and rotational-mechanical systems in time-, frequency-domain, and transfer function representations
- Nonlinearities
- ► Linearization of nonlinear systems in time-, frequency-domain, and transfer function representations

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Chapter outline

- 2 Modeling in the frequency domain
 - 2.1 Introduction
 - 2.2 Laplace transform review
 - 2.3 The transfer function
 - 2.4 Electrical network transfer functions
 - 2.5 Translational mechanical system transfer functions
 - 2.6 Rotational mechanical system transfer functions
 - 2.7 Transfer functions for systems with gears
 - 2.8 Electromechanical system transfer functions
 - 2.9 Electric circuit analogs
 - 2.10 Nonlinearities
 - 2.11 Linearization

2 Modeling in the frequency domain

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The Laplace transform definitions, [1, p. 35]

- 2.9 Electric circuit analogs
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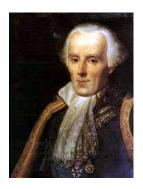
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2.2 Laplace transform review

History interlude

Pierre-Simon Laplace

- **▶** 1749 1827
- ► French mathematician and astronomer
- ► Pioneered the *Laplace* transform
- ► AKA French Newton
- "...all the effects of nature are only mathematical results of a small number of immutable laws."
- "What we know is little, and what we are ignorant of is immense."



Laplace transform

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Inverse Laplace transform

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds$$
$$= f(t)u(t)$$

where

$$s = \sigma + j\omega$$

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2 Modeling in the frequency domain

2.2 Laplace transform review

Laplace transform table, [1, p. 36]

f(t)	F(s)
$\delta(t)$	1
u(t)	$\frac{1}{s}$
tu(t)	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin(\omega t)u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin(\omega t)u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

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Modeling in the frequency domain

2.2 Laplace transform review

Laplace transform theorems, [1, p. 37]

Some basic algebraic operations, such as multiplication by exponential functions or shifts have simple counterparts in the Laplace domain

Theorem (Frequency shift)

$$\mathcal{L}[e^{-at}f(t)] = F(s+a)$$

Theorem (Time shift)

$$\mathcal{L}[f(t-T)] = e^{-sT}F(s)$$

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2 Modeling in the frequency domain

2.2 Laplace transform review

Laplace transform theorems, [1, p. 37]

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Theorem (Linearity)

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1 F_1(s) + c_2 F_2(s)$$

Theorem (Scaling)

$$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Theorem (Differentiation)

Laplace transform theorems, [1, p. 37]

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - \sum_{k=1}^n s^{n-k} \frac{d^{k-1} f}{dt^{k-1}} (0-)$$

Examples

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$$

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2.2 Laplace transform review

Laplace transform theorems, [1, p. 37]

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Laplace transform theorems, [1, p. 37]

Theorem (Integration)

$$\mathcal{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$$

Theorem (Final value)

$$[f(\infty)] = \lim_{s \to 0} \quad sF(s)$$

To yield correct finite results, all roots of the denominator of F(s) must have negative real parts, and no more than one can be at the origin.

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2 Modeling in the frequency domain

2.2 Laplace transform review

Laplace transform theorems, [1, p. 37]

Theorem (Initial value)

$$[f(0+)] = \lim_{s \to \infty} sF(s)$$

To be valid, f(t) must be continuous or have a step discontinuity at t=0, i.e., no impulses or their derivatives at t=0.

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2.2 Laplace transform review

Partial fraction expansion, [1, p. 37]

To find the inverse Laplace transform of a complicated function, we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term

$$F(s) = \frac{N(s)}{D(s)}$$

How F(s) can be expanded is governed by the relative order between ${\cal N}(s)$ and ${\cal D}(s)$

- 1. $\mathcal{O}(N(s)) < \mathcal{O}(D(s))$
- 2. $\mathcal{O}(N(s)) \geq \mathcal{O}(D(s))$

and the type of roots of $D(\boldsymbol{s})$

- 1. Real and distinct
- 2. Real and repeated
- 3. Complex or imaginary

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The transfer function, [1, p. 44]

General n-th order, linear, time-invariant differential equation

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \dots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \dots + b_0 r(t)$$

Under the assumption that *all initial conditions are zero* the transfer function (TF) from input, c(t), to output, r(t), i.e., the ratio of the output transform, C(s), divided by the input transform, R(s) is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}$$

Also, the output transform, C(s) can be written as

$$C(s) = R(s)G(s) \\$$

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2 Modeling in the frequency domain

2.4 Electrical network TFs

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2 Modeling in the frequency domain 2.4 Electrical network TF

Electrical network TFs, [1, p. 47]

Table: Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
——————————————————————————————————————	$v(t) = \frac{1}{C} \int_0^1 i(\tau) d\tau$	$i(t) = C \frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
-\\\\- Resistor	v(t)=Ri(t)	$i(t) = \frac{1}{R}v(t)$	$v(t)=R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L \frac{di(t)}{dt}$	$i(t) = \frac{1}{L} \int_0^1 v(\tau) d\tau$	$v(t) = L \frac{d^2 q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

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Electrical network TFs, [1, p. 59]

Electrical network TFs, [1, p. 48]

Example (Resistor-inductor-capacitor (RLC) system)

- ▶ Problem: Find the TF relating the capacitor voltage, $V_C(s)$, to the input voltage, V(s)
- ► Solution: On board

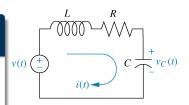


Figure: RLC system

Example (Inverting operational amplifier system)

- ▶ Problem: Find the TF relating the output voltage, $V_o(s)$, to the input voltage $V_i(s)$
- ► *Solution:* On board

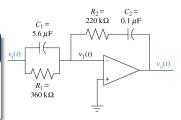


Figure: Inverting operational amplifier system

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2 Modeling in the frequency domain 2.5 Translational mechanical system

Translational mechanical system TFs, [1, p. 61]

Table: Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
$f(t) = K \int_0^t v(\tau) d\tau$	f(t) = Kx(t)	К
$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_{\nu}s$
$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2 x(t)}{dt^2}$	Ms^2
	$f(t) = K \int_0^t v(\tau) d\tau$ $f(t) = f_v v(t)$	$f(t) = K \int_0^t v(\tau) d\tau \qquad \qquad f(t) = Kx(t)$ $f(t) = f_v v(t) \qquad \qquad f(t) = f_v \frac{dx(t)}{dt}$

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2 Modeling in the frequency domain 2.5 Translational mechanical system

Translational mechanical system TFs, [1, p. 63]

Example (Translational inertia-spring-damper system)

- ▶ *Problem:* Find the TF relating the position, X(s), to the input force, F(s)
- Solution: On board

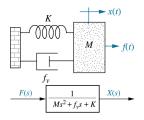


Figure: Physical system; block diagram

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2 Modeling in the frequency domain

2.6 Rotational mechanical system TFs

Rotational mechanical system TFs, [1, p. 69]

Table: Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
Spring $T(t) \theta(t)$	$T(t) = K \int_0^t \omega(\tau) d\tau$	$T(t) = K\theta(t)$	K
Viscous $T(t)$ $\theta(t)$ damper	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
Inertia $T(t) \theta(t)$	$T(t) = J \frac{d\omega(t)}{dt}$	$T(t) = J \frac{d^2\theta(t)}{dt^2}$	Js^2

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Rotational mechanical system TFs, [1, p. 63]

Example (Rotational inertia-spring-damper system) Problem: Find the TF relating the position, $\Theta_2(s)$, to the

input torque, T(s)

► *Solution:* On board

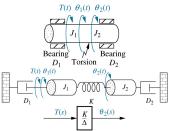


Figure: Physical system; schematic; block diagram

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2 Modeling in the frequency domain

2.10 Nonlinearities

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2 Modeling in the frequency domain 2.10 Nonlineari

Common physical nonlinearities found in nonlinear (NL) systems

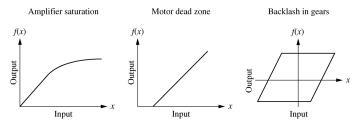


Figure: Some physical nonlinearities

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2 Modeling in the frequency domain 2.11 Linearization

Linearization, [1, p. 89]

Motivation

▶ Must linearize a NL system into a LTI DE before we can find a TF

Linearization procedure

- 1. Recognize the NL component and write the NL DE
- 2. Linearize the NL DE into an LTI DE
- 3. Laplace transform of LTI DE assuming zero initial conditions
- 4. Separate input and output variables
- 5. Form the TF

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General linearization via Taylor series expansion

Linearization, [1, p. 89]

1^{st} -order linearization

- ightharpoonup Output, f(x)
- ightharpoonup Input, x
- ightharpoonup Operating at point A, $[x_0, f(x_0)]$
- $\,\blacktriangleright\,$ Small changes in the input can be related to changes in the output about the point by way of the slope of the curve, m_a , at point A

$$[f(x) - f(x_0)] \approx m_a(x - x_0)$$
$$\delta f(x) \approx m_a \delta x$$
$$f(x) \approx f(x_0) + m_a \delta x$$

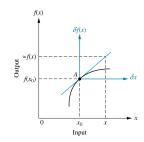


Figure: Linearization about point \boldsymbol{A}

Linearization, [1, p. 89]

$$f(x) = f(x_0) + \frac{df}{dx}|_{x=x_0} \frac{(x-x_0)}{1!} + \frac{d^2f}{dx^2}|_{x=x_0} \frac{(x-x_0)^2}{2!} + \dots$$

For small excursions of x from x_0 , we can neglect higher-order terms. The resulting approximation yields a straight-line relationship between the change in f(x) and the excursion away from x_0 . Neglecting higher-order terms yields

$$f(x) - f(x_0) pprox rac{df}{dx}|_{x=x_0}(x-x_0)$$
 or $\nabla f = rac{\delta f}{\delta x} pprox m|_{x=x_0}$

which is a linear relationship between $\delta f(x)$ and δx for small excursions away from x_0 .

Linearization, [1, p. 92]

Example (NL electrical system)

- ▶ *Problem:* Find the TF relating the inductor voltage, $V_L(s)$, to the input voltage, V(s). The NL resistor voltage-current relationship is defined by $i_r = 2e^{0.1v_r}$, where i_r and v_r are the resistor current and voltage, respectively. Also the input voltage, v, is a small-signal source.
- Solution: On board

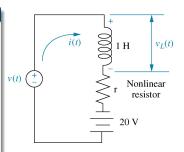


Figure: NL electrical system

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