EE C128 / ME C134 – Feedback Control Systems

Lecture – Chapter 8 – Root Locus Techniques

Alexandre Bayen

Department of Electrical Engineering & Computer Science
University of California Berkeley

September 10, 2013

Lecture abstract

Topics covered in this presentation

- What is root locus
- System analysis via root locus
- How to plot root locus

Lecture outline

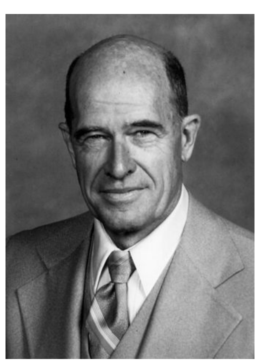
- 8 Root Locus Techniques
  - 8.1 Introduction
  - 8.2 Defining the root locus
  - 8.3 Properties of the root locus
  - 8.4 Sketching the root locus
  - 8.5 Refining the sketch
  - 8.6 An example
  - 8.7 Transient response design via gain adjustments
  - 8.8 Generalized root locus
  - 8.9 Root locus for positive-feedback systems
  - 8.10 Pole sensitivity

- 8 Root Locus Techniques
  - 8.1 Introduction
  - 8.2 Defining the root locus
  - 8.3 Properties of the root locus
  - 8.4 Sketching the root locus
  - 8.5 Refining the sketch
  - 8.6 An example
  - 8.7 Transient response design via gain adjustments
  - 8.8 Generalized root locus
  - 8.9 Root locus for positive-feedback systems
  - 8.10 Pole sensitivity

History interlude

Walter Richard Evans

- 1920 – 1999
- American control theorist
- 1948 – Inventor of the *root locus* method
- 1988 – Richard E. Bellman Control Heritage Award



Definitions, [1, p. 388]

Root locus (RL)

- Uses the *poles and zeros of the OL TF* (product of the forward path TF and FB path TF) to analyze and design the *poles of a CL TF* as a system (plant or controller) parameter, K, that shows up as a gain in the OL TF is varied
- Graphical representation of
  - Stability (CL poles)
    - Range of stability, instability, & marginal stability
  - Transient response
    - T_r, T_s, & $\%OS$
- Solutions for systems of order > 2

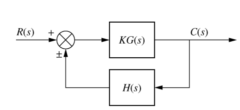


Figure: ±FB system

The control system problem, [1, p. 388]

- OL TF

$$KG(s)H(s)$$

- OL TF poles unaffected by the *one* system gain, K

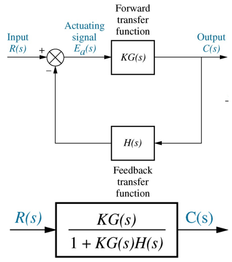


Figure: a. -FB CL system; b. equivalent function

The control system problem, [1, p. 388]

- Forward TF

$$G(s) = \frac{N_G(s)}{D_G(s)}$$

- Feedback TF

$$H(s) = \frac{N_H(s)}{D_H(s)}$$

- -FB CL TF

$$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}$$

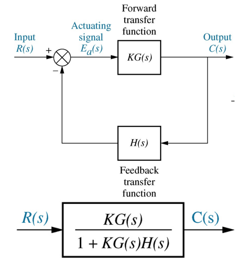


Figure: a. -FB CL system, b. equivalent function

Vector representation of complex numbers, [1, p. 388]

- Cartesian, $\sigma + j\omega$
- Polar, $M\angle\theta$
  - Magnitude, M
  - Angle, θ
- Function, $F(s)$
  - Example, $(s + a)$
    - Vector from the zero, a, of the function to the point s

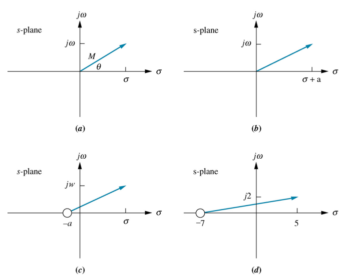


Figure: Vector representation of complex numbers: a. $s = \sigma + j\omega$, b. $(s + a)$; c. alternate representation of $(s + a)$, d. $(s + 7)|_{s \to 5+j2}$

Vector representation of complex numbers, [1, p. 390]

- Function, $F(s)$
  - Complicated

$$F(s) = \frac{\prod_{i=1}^{m}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = \frac{\prod \text{numerator's complex factors}}{\prod \text{denominator's complex factors}}$$

- Magnitude

$$M = \frac{\prod \text{zero lengths}}{\prod \text{pole lengths}} = \frac{\prod_{i=1}^{m}|(s + z_i)|}{\prod_{j=1}^{n}|(s + p_j)|}$$

- Angle

$$\theta = \sum \text{zero angles} - \sum \text{pole angles} = \sum_{i=1}^{m}\angle(s + z_i) - \sum_{i=j}^{n}\angle(s + p_j)$$

■ 8 Root Locus Techniques
- 8.1 Introduction
- **8.2 Defining the root locus**
- 8.3 Properties of the root locus
- 8.4 Sketching the root locus
- 8.5 Refining the sketch
- 8.6 An example
- 8.7 Transient response design via gain adjustments
- 8.8 Generalized root locus
- 8.9 Root locus for positive-feedback systems
- 8.10 Pole sensitivity

■ 8 Root Locus Techniques
- 8.1 Introduction
- 8.2 Defining the root locus
- **8.3 Properties of the root locus**
- 8.4 Sketching the root locus
- 8.5 Refining the sketch
- 8.6 An example
- 8.7 Transient response design via gain adjustments
- 8.8 Generalized root locus
- 8.9 Root locus for positive-feedback systems
- 8.10 Pole sensitivity

-FB CL poles, [1, p. 394]

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

The angle of the complex number is an *odd* multiple of $180°$

$$KG(s)H(s) = -1 = 1\angle(2k+1)180°$$
$$k = 0, \pm1, \pm2, \pm3, \ldots$$

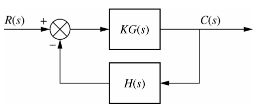


Figure: -FB system

The system gain, K, satisfies *magnitude criterion*

$$|KG(s)H(s)| = 1$$

angle criterion

$$\angle KG(s)H(s) = (2k+1)180°$$

and thus

$$K = \frac{1}{|G(s)||H(s)|}$$

- 8 Root Locus Techniques
  - 8.1 Introduction
  - 8.2 Defining the root locus
  - 8.3 Properties of the root locus
  - 8.4 Sketching the root locus
  - 8.5 Refining the sketch
  - 8.6 An example
  - 8.7 Transient response design via gain adjustments
  - 8.8 Generalized root locus
  - 8.9 Root locus for positive-feedback systems
  - 8.10 Pole sensitivity

Basic rules for sketching -FB RL, [1, p. 397]

- *Number of branches:* Equals the number of CL poles
- *Symmetry:* About the real axis
- *Real-axis segments:* On the real axis, for $K > 0$, the RL exists to the left of an *odd* number of real-axis, finite OL poles and/or finite OL zeros
- *Starting and ending points:* The RL begins at the finite & infinite poles of G(s)H(s) and ends at the finite & infinite zeros of $G(s)H(s)$

Basic rules for sketching -FB RL, [1, p. 397]

- *Behavior at ∞:* The RL approaches straight lines as asymptotes as the RL approaches ∞. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a, and angle, θ_a, as follows

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}$$

where $k = 0, \pm1, \pm2, \pm3, \ldots$ and the angle is given in radians with respect to the positive extension of the real-axis

Example, [1, p. 400]

Example (-FB RL with asymptotes)

- *Problem:* Sketch the RL
- *Solution:* On board

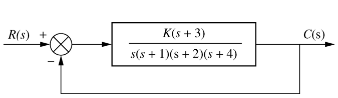


Figure: System

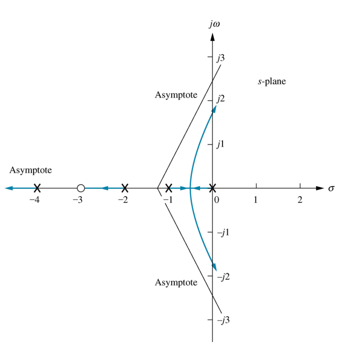


Figure: RL & asymptotes for system

- 8 Root Locus Techniques
  - 8.1 Introduction
  - 8.2 Defining the root locus
  - 8.3 Properties of the root locus
  - 8.4 Sketching the root locus
  - 8.5 Refining the sketch
  - 8.6 An example
  - 8.7 Transient response design via gain adjustments
  - 8.8 Generalized root locus
  - 8.9 Root locus for positive-feedback systems
  - 8.10 Pole sensitivity

Additional rules for refining a RL sketch, [1, p. 402]

- *Real-axis breakaway & break-in points:* At the breakaway or break-in point, the branches of the RL form an angle of $180°/n$ with the real axis, where n is the number of CL poles arriving at or departing from the single breakaway or break-in point on the real-axis.
- *The $j\omega$-axis crossings:* The $j\omega$-crossing is a point on the RL that separates the stable operation of the system from the unstable operation.
- *Angles of departure & arrival:* The value of ω at the axis crossing yields the frequency of oscillation, while the gain, K, at the $j\omega$-axis crossing yields the maximum or minimum positive gain for system stability.
- *Plotting & calibrating the RL:* All points on the RL satisfy the angle criterion, which can be used to solve for the gain, K, at any point on the RL.

Differential calculus procedure, [1, p. 402]

Procedure

- *Maximize & minimize the gain, K, using differential calculus:* The RL breaks away from the real-axis at a point where the gain is maximum and breaks into the real-axis at a point where the gain is minimum. For all points on the RL

$$K = -\frac{1}{G(s)H(s)}$$

For points along the real-axis segment of the RL where breakaway and break-in points could exist, $s = \sigma$. Differentiating with respect to σ and setting the derivative equal to zero, results in points of maximum and minimum gain and hence the breakaway and break-in points.

Transition procedure, [1, p. 402]

Procedure

- Eliminates the need to differentiate. Breakaway and break-in points satisfy the relationship

$$\sum_{i=1}^{m} \frac{1}{\sigma + z_i} = \sum_{j=1}^{n} \frac{1}{\sigma + p_j}$$

where z_i and p_i are the negative of the zero and pole values, respectively, of $G(s)H(s)$.

The $j\omega$-crossings, [1, p. 405]

Procedures for finding $j\omega$-crossings

- Using the Routh-Hurwitz criterion, forcing a row of zeros in the Routh table will yield the gain; going back one row to the even polynomial equation and solving for the roots yields the frequency at the imaginary-axis crossing.
- At the $j\omega$-crossing, the sum of angles from the finite OL poles & zeros must add to $(2k+1)180°$. Search the $j\omega$-axis for a point that meets this angle condition.

Angles of departure & arrival, [1, p. 407]

The RL departs from complex, OL poles and arrives at complex, OL zeros

- Assume a point ϵ close to the complex pole or zero. Add all angles drawn from all OL poles and zeros to this point. The sum equals $(2k+1)180°$. The only unknown angle is that drawn from the ϵ close pole or zero, since the vectors drawn from all other poles and zeros can be considered drawn to the complex pole or zero that is ϵ close to the point. Solving for the unknown angle yields the angle of departure or arrival.

Plotting & calibrating the RL, [1, p. 410]

Search a given line for a point yielding

$$\sum \text{zero angles} - \sum \text{pole angles} = (2k+1)180°$$

or

$$\angle G(s)H(s) = (2k+1)180°$$

The gain at that point on the RL satisfies

$$K = \frac{1}{|G(s)H(s)|} = \frac{\prod \text{finite pole lengths}}{\prod \text{finite zero lengths}}$$

- 8 Root Locus Techniques
  - 8.1 Introduction
  - 8.2 Defining the root locus
  - 8.3 Properties of the root locus
  - 8.4 Sketching the root locus
  - 8.5 Refining the sketch
  - 8.6 An example
  - 8.7 Transient response design via gain adjustments
  - 8.8 Generalized root locus
  - 8.9 Root locus for positive-feedback systems
  - 8.10 Pole sensitivity

Example, [1, p. 412]

Example (-FB RL & critical points)

- *Problem:* Sketch RL & find
  - $\zeta = 0.45$ line crossing
  - $j\omega$-axis crossing
  - The breakaway point
  - The range of stable K
- *Solution:* On board

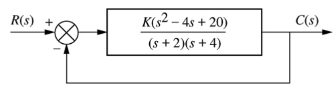


Figure: System

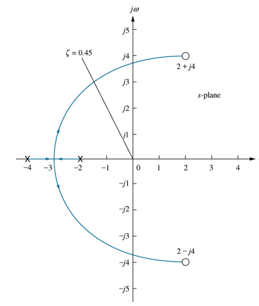


Figure: RL

- 8 Root Locus Techniques
  - 8.1 Introduction
  - 8.2 Defining the root locus
  - 8.3 Properties of the root locus
  - 8.4 Sketching the root locus
  - 8.5 Refining the sketch
  - 8.6 An example
  - 8.7 Transient response design via gain adjustments
  - 8.8 Generalized root locus
  - 8.9 Root locus for positive-feedback systems
  - 8.10 Pole sensitivity

Conditions justifying a 2^{nd}-order approximation, [1, p. 415]

- Higher-order poles are much farther (rule of thumb: $> 5\times$) into the LHP than the dominant 2^{nd}-order pair of poles.
- CL zeros near the CL 2^{nd}-order pole pair are nearly canceled by the close proximity of higher-order CL poles.
- CL zeros not canceled by the close proximity of higher-order CL poles are far removed from the CL 2^{nd}-order pole pair.

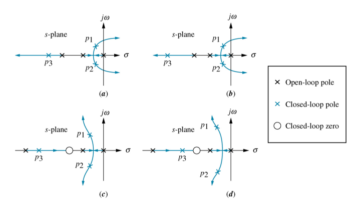


Figure: Making 2^{nd}-order approximation

Higher-order system design, [1, p. 416]

Procedure

1. Sketch RL
2. Assume the system is a 2^{nd}-order system without any zeros and then find the gain to meet the transient response specification
3. Justify your 2^{nd}-order assumptions
4. If the assumptions cannot be justified, your solution will have to be simulated in order to be sure it meets the transient response specification. It is a good idea to simulate all solutions, anyway

- 8 Root Locus Techniques
  - 8.1 Introduction
  - 8.2 Defining the root locus
  - 8.3 Properties of the root locus
  - 8.4 Sketching the root locus
  - 8.5 Refining the sketch
  - 8.6 An example
  - 8.7 Transient response design via gain adjustments
  - 8.8 Generalized root locus
  - 8.9 Root locus for positive-feedback systems
  - 8.10 Pole sensitivity

Example, [1, p. 419]

Example (-FB RL with a parameter pole)

- *Problem:* Create an equivalent system whose denominator is

$$1 + p_1 G(s)H(s)$$

  and sketch the RL
- *Solution:* On board

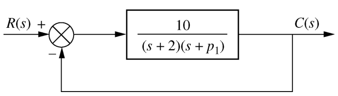


Figure: System

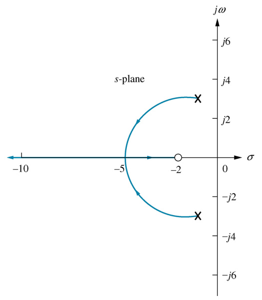


Figure: RL

- 8 Root Locus Techniques
  - 8.1 Introduction
  - 8.2 Defining the root locus
  - 8.3 Properties of the root locus
  - 8.4 Sketching the root locus
  - 8.5 Refining the sketch
  - 8.6 An example
  - 8.7 Transient response design via gain adjustments
  - 8.8 Generalized root locus
  - 8.9 Root locus for positive-feedback systems
  - 8.10 Pole sensitivity

+FB CL poles, [1, p. 394]

$$T(s) = \frac{KG(s)}{1 - KG(s)H(s)}$$

The angle of the complex number is an *even* multiple of 180°

$$KG(s)H(s) = 1 = 1\angle k360^\circ$$
$$k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

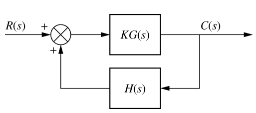


Figure: +FB system

The system gain, K, satisfies *magnitude criterion*

$$|KG(s)H(s)| = -1$$

angle criterion

$$\angle KG(s)H(s) = k360^\circ$$

and thus

$$K = \frac{1}{|G(s)||H(s)|}$$

Basic rules for sketching +FB RL, [1, p. 421]

- *Number of branches:* Equals the number of CL poles *(same as -FB)*
- *Symmetry:* About the real axis *(same as -FB)*
- *Real-axis segments:* On the real axis, for $K > 0$, the RL exists to the left of an *even* number of real-axis, finite OL poles and/or finite OL zeros
- *Starting and ending points:* The RL begins at the finite & infinite poles of G(s)H(s) and ends at the finite & infinite zeros of $G(s)H(s)$ *(same as -FB)*

Basic rules for sketching +FB RL, [1, p. 422]

- *Behavior at ∞:* The RL approaches straight lines as asymptotes as the RL approaches ∞. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a, and angle, θ_a, as follows

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$\theta_a = \frac{k2\pi}{\#\text{finite poles} - \#\text{finite zeros}}$$

  where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$ and the angle is given in radians with respect to the positive extension of the real-axis.

- 8 Root Locus Techniques
  - 8.1 Introduction
  - 8.2 Defining the root locus
  - 8.3 Properties of the root locus
  - 8.4 Sketching the root locus
  - 8.5 Refining the sketch
  - 8.6 An example
  - 8.7 Transient response design via gain adjustments
  - 8.8 Generalized root locus
  - 8.9 Root locus for positive-feedback systems
  - 8.10 Pole sensitivity

Definitions, [1, p. 424]

Root sensitivity

- The ratio of the fractional change in a CL pole to the fractional change in a system parameter, such as a gain.

Sensitivity of a CL pole, s, to gain, K

$$S_{s:K} = \frac{K}{s}\frac{\delta s}{\delta K}$$

Approximated as

$$\Delta s = s(S_{s:K})\frac{\Delta s}{K}$$

where $\frac{\delta s}{\delta K}$ is found by differentiating the CE with respect to K

Example, [1, p. 425]

Example (root sensitivity of a CL system to gain variations)

- *Problem:* Find the root sensitivity of the system at $s = -5 + j5$ (for which $K = 50$) and calculate the change in the pole location for a 10% change in K
- *Solution:* On board

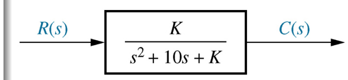


Figure: System

Bibliography

Norman S. Nise. *Control Systems Engineering*, 2011.